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Asymptotic Stability in Linear Thermoviscoelasticity*

CARLOS B. NAVARRO

Instituto de Matematica Aplicada, Escuela Tecnica Superior de Arquitectura, Universidad Politecnica de Barcelona, Avda. Generalisimo 649, Barcelona, Spain

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1. INTRODUCTION

The investigation of the stability of processes in materials with hereditary response and the relation of stability to thermodynamics are currently receiving considerable attention [1–6]. An important example of hereditary response is the constitutive relation of isothermal linear viscoelasticity, where the dependence of the stress on the history of the infinitesimal strain tensor is of the Boltzmann type [7]. The existence of a genuine memory induces a damping mechanism, and asymptotic stability is to be expected.

In the present paper we study the existence, uniqueness, and asymptotic behavior of solutions to the evolution equations for a body composed of an inhomogeneous anisotropic linear thermoviscoelastic material, whose constitutive equations are an extension of linear viscoelasticity to the nonisothermal situation. The histories of the independent kinematic variables—displacement and temperature difference—are supposed given up to some time $t = 0$. Homogeneous boundary conditions for the displacement and temperature difference fields are assumed, as is the absence of any longrange mechanical or thermal action from the outside world at all positive times. The main objective is to prove asymptotic stability.

After the introductory Section 2, in which we collect the necessary mathematical prerequisites, we proceed in Section 3 to set down the evolution equations, together with some restrictions on the material properties of the body imposed by the Clausius–Duhem inequality. Those restrictions motivate some of the assumptions made subsequently. Next, the boundary-initial value problem is formulated in a classical sense.

Following a method based on the work of Visik and Ladyzenskaya [9] and Lions [10], and applied by Dafermos [11] in his study of an abstract Volterra equation, uniqueness and existence of a generalized solution is proved in Section 4.

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Then we investigate the smoothness in time of the generalized solution by introducing spaces of fading memory type for the initial histories [12].

In Section 5 the asymptotic stability of solutions is studied. A suitable functional is defined and its monotonicity properties are investigated. This functional is essentially a free energy functional of the kind discussed by Coleman [4] and Coleman and Dill [5, 6]. Finally, a theorem which accurately describes the behavior of solutions as $t \to \infty$ is proved with the aid of the functional. The set of conditions on the material sufficient to guarantee asymptotic stability, although rather stringent, turns out to be realistic. Using a modified form of the theory of dynamical systems constructed by Hale [13, 14], analogous results are obtained in [8] for the purely mechanical case.

2. Mathematical Prerequisites

The underlying space will be the three-dimensional Euclidean point space R^3 in which the body is embedded. A typical element of R^3 is the point $\mathbf{x}$, and we shall use the term vector for elements of the associated vector space V. The processes of the body are described in $R^3 \times R^1$, $(\mathbf{x}, t)$ being a typical point where t is the time variable.

We write $\mathbf{u} \cdot \mathbf{v}$ for the scalar product of the vectors $\mathbf{u}$ and $\mathbf{v}$, and $|\mathbf{u}| = (\mathbf{u} \cdot \mathbf{u})^{1/2}$ for the magnitude of $\mathbf{u}$.

The term tensor stands for any linear transformation from the space V into a finite-dimensional inner product space U.

A second-order tensor is a linear transformation from V into itself. The collection of all second-order tensors can be regarded as a nine-dimensional vector space having the scalar product $\mathbf{M}_1 \cdot \mathbf{M}_2 = \operatorname{trace} \mathbf{M}_1 \mathbf{M}_2{}^T$, where $\mathbf{M}_2{}^T$, the transpose of $\mathbf{M}_2$, is defined by the condition $\mathbf{u} \cdot \mathbf{M}_2 \mathbf{v} = \mathbf{v} \cdot \mathbf{M}_2{}^T \mathbf{u}$ holding for every pair of vectors $\mathbf{u}$, $\mathbf{v}$. The magnitude of $\mathbf{M}$ is given by $|\mathbf{M}| = (\mathbf{M} \cdot \mathbf{M})^{1/2}$.

A fourth-order tensor can be considered as a linear transformation of the space of second-order tensors into itself. The transpose, $\mathbf{G}^T$, of a fourth-order tensor $\mathbf{G}$ is defined by the condition $\mathbf{M}_1 \cdot \mathbf{G}\mathbf{M}_2 = \mathbf{M}_2 \cdot \mathbf{G}^T \mathbf{M}_1$ holding for every pair of second-order tensors $\mathbf{M}_1$, $\mathbf{M}_2$. The magnitude, $|\mathbf{G}|$, of $\mathbf{G}$ is defined by

$$|\mathbf{G}| = \sup_{|\mathbf{M}|=1} |\mathbf{G}\mathbf{M}| .$$

We shall deal with scalar, vector, and tensor fields on subsets $B \times E \subset R^3 \times R^1$, by which we mean functions that assign to each $(\mathbf{x}, t) \in B \times E$ a scalar, vector, or tensor, respectively.

Let $B \times E$ be open and let Φ be a mapping from $B \times E$ into U, a finite-dimensional inner product space. We hold $t \in E$ fixed. We say that Φ is differentiable at $\mathbf{x} \in B$ if there exists a linear function $\mathbf{f}\colon V \to U$ such that

$$\Phi(\mathbf{x}', t) - \Phi(\mathbf{x}, t) = \mathbf{f}\{\mathbf{x}' - \mathbf{x}\} + o(|\mathbf{x}' - \mathbf{x}|) \qquad \text{as} \qquad \mathbf{x}' \to \mathbf{x}$$

and we call $\mathbf{f} = \nabla\Phi(\mathbf{x}, t)$ the gradient of Φ at $\mathbf{x}$ at time t. We can define in the same manner the nth gradient, $\nabla^{(n)}\Phi$, of Φ at $\mathbf{x}$ holding t fixed.

If $\mathbf{u}$ is a vector field differentiable at $\mathbf{x} \in B$, the divergence of $\mathbf{u}$ at $\mathbf{x}$ is the scalar $\operatorname{div} \mathbf{u}(\mathbf{x}, t) = \operatorname{trace} \nabla\mathbf{u}(\mathbf{x}, t)$. The divergence at $\mathbf{x}$ of a second-order tensor field $\mathbf{M}$, differentiable at $\mathbf{x}$, is the unique vector with the property $[\operatorname{div} \mathbf{M}(\mathbf{x}, t)] \cdot \mathbf{c} = \operatorname{div}[\mathbf{M}^T(\mathbf{x}, t)\, \mathbf{c}]$ for every fixed vector $\mathbf{c}$.

Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be an orthonormal basis, let (x_1, x_2, x_3) be the components of $\mathbf{x}$ and let $\Phi_{i_1 \cdots i_k}$ be the components of the kth order tensor field Φ. The derivatives of Φ with respect to the space variables are defined by the components

$$\frac{\partial^l \Phi_{i_1 \cdots i_k}}{\partial x_{j_1} \cdots \partial x_{j_l}} = \mathbf{e}_{i_1} \cdot \{[(((\nabla^{(l)}\Phi \mathbf{e}_{j_l})\, \mathbf{e}_{j_{l-1}}) \cdots \mathbf{e}_{j_1})]\, \mathbf{e}_{i_k} \cdots\}\, \mathbf{e}_{i_2}$$

of the tensor $\nabla^{(l)}\Phi$ at $\mathbf{x}$ holding t fixed.

We say that Φ is differentiable in B if Φ is differentiable at every point of B. By $C^n(B)$ we shall denote the class of fields differentiable up to the nth order in B and with $\nabla^{(l)}\Phi$, $l = 0, 1, \ldots, n$, continuous on $\bar{B}$, the closure of B. By $C^\infty(B)$ we debote $\bigcap_{n=0}^{\infty} C^n(B)$ and if support $\Phi = \bar{S}$ is compact, where $S = \{\mathbf{x} \mid \Phi(\mathbf{x}) \neq 0\}$, and $\bar{S} \subset B$ we have the subclass $C_0^\infty(B)$ of test functions.

We write $\dot{\Phi}$, $\ddot{\Phi}, \ldots, \Phi^{(n)}$, for the first, second,..., nth derivative of Φ with respect to time t holding $\mathbf{x}$ fixed. If the maps Φ are elements of a Banach space $\mathscr{D}(B)$, and E is a time interval with closure $\bar{E}$, we denote by $C^n(\bar{E}; \mathscr{D}(B))$ the class of functions Φ with domain $\bar{E}$ and range in $\mathscr{D}(B)$, which possess on E time derivatives of order up to n, continuous on $\bar{E}$.

Now let B be an open and bounded set of R^3. We shall denote by $W_2{}^n(B)$ the Hilbert space completion of $C^n(B)$ under the norm $\|\cdot\|_n$ defined by

$$\| \Phi \|_n^2 = \sum_{l=0}^{n} \int_B | \nabla^{(l)}\Phi |^2\, d\mathbf{x},$$

where $d\mathbf{x} = dx_1\, dx_2\, dx_3$. When the fields Φ are scalar fields we shall use the notation $C^n(B)$, $W_2{}^n(B)$, and $\mathbf{C}^n(B)$, $\mathbf{W}_2{}^n(B)$ when we deal with vector fields. $\mathring{W}_2{}^n(B)$ $[\mathring{\mathbf{W}}_2{}^n(B)]$ will be the Hilbert space obtained by the closure of $C_0{}^n(B)$ $[\mathbf{C}_0{}^n(B)]$ with respect to the norm of $W_2{}^n(B)$ $[\mathbf{W}_2{}^n(B)]$. Poincaré's inequality implies that

$$\int_B | \nabla^{(n)}\Phi |^2\, d\mathbf{x}$$

is a norm equivalent to $\|\cdot\|_n$ for the space $\mathring{W}_2{}^n(B)$.

An integrable function Φ on B is said to have a weak derivative, denoted by the same symbol as the corresponding ordinary derivative, if

$$\int_B \varphi \frac{\partial^l \Phi}{\partial x_{j_1} \cdots \partial x_{j_l}}\, d\mathbf{x} = (-1)^l \int_B \Phi \frac{\partial^l \varphi}{\partial x_{j_1} \cdots \partial x_{j_l}}\, d\mathbf{x}$$

for all functions $\varphi \in C_0^\infty(B)$. In general, $W_2^n(B)$ coincides with the class of functions in $L_2(B)$ having weak derivatives in $L_2(B)$ of order up to n.[1]

3. Boundary-Initial Value Problem

In this section we motivate the constitutive equations defining the linear thermoviscoelastic material, as well as some restrictions on them induced by the entropy production inequality. Then, we formulate the special boundary-initial value problem of the coupled dynamic theory which we develop in a general sense in the following sections.

Consider the body to be identified with the region it occupies in a fixed reference configuration which we take as a natural state, namely, a state with zero stress and constant base temperature θ_0, strictly positive. Let $\mathbf{x}$ be the position of a material point at time t, let $\mathbf{u}(\mathbf{x}, t)$ be the displacement and let $\Theta(\mathbf{x}, t) - \theta_0 = \theta(\mathbf{x}, t)$ be the temperature difference from θ_0.

We postulate a specific Helmholtz free energy functional, $\Psi(\mathbf{x}, t)$, depending upon both displacement and temperature difference history in a quadratic manner:[2]

$$\begin{aligned}\rho\Psi(t) = {} & \frac{1}{2}\nabla\mathbf{u}(t)\cdot\mathbf{G}(\infty)\nabla\mathbf{u}(t) - \theta(t)\,\mathbf{L}(\infty)\cdot\nabla\mathbf{u}(t) - \frac{1}{2}\rho\frac{c(\infty)}{\theta_0}\theta^2(t) \\ & - \tfrac{1}{2}\int_{-\infty}^{t}[\nabla\mathbf{u}(t) - \nabla\mathbf{u}(\tau)]\cdot\dot{\mathbf{G}}(t-\tau)[\nabla\mathbf{u}(t) - \nabla\mathbf{u}(\tau)]\,d\tau \\ & + \int_{-\infty}^{t}[\theta(t) - \theta(\tau)]\,\dot{\mathbf{L}}(t-\tau)\cdot[\nabla\mathbf{u}(t) - \nabla\mathbf{u}(\tau)]\,d\tau \\ & + \tfrac{1}{2}\rho/\theta_0\int_{-\infty}^{t}\dot{c}(t-\tau)[\theta(t) - \theta(\tau)]^2\,d\tau,\end{aligned} \tag{3.1}$$

where $\rho(\mathbf{x})$ is the mass density field in the natural state. The material properties $\mathbf{G}(\mathbf{x}, s)$, $\mathbf{L}(\mathbf{x}, s)$, and $c(\mathbf{x}, s)$, $s \geqslant 0$, are the relaxation tensor fields of fourth, second, and zero order, respectively. We assume that they are functions of class C^2 on their domain of definition $[0, \infty)$. By $\mathbf{G}(\mathbf{x}, \infty)$, $\mathbf{L}(\mathbf{x}, \infty)$, $c(\mathbf{x}, \infty)$ we denote the limits

$$\lim_{s\to\infty}\mathbf{G}(\mathbf{x}, s), \qquad \lim_{s\to\infty}\mathbf{L}(\mathbf{x}, s), \qquad \lim_{s\to\infty}c(\mathbf{x}, s),$$

which we can call the equilibrium elasticity modulus, equilibrium stress-tempe-

[1] A comprehensive study of the Sobolev spaces W_p^n can be found in [15] for $p = 2$, and in [16] for the general case $p \geqslant 1$.

[2] The dependence on $\mathbf{x}$ is omitted for convenience.

rature tensor, and equilibrium specific heat. We assume that the symmetry conditions[3]

$$\mathbf{G}(s) = \mathbf{G}^T(s), \qquad \mathbf{L}(s) = \mathbf{L}^T(s) \tag{3.2}$$

are satisfied for all $s \geqslant 0$.

The local statement of the energy balance law is given by

$$\rho r - \rho[\dot{\Psi} + \dot{\theta}\eta + \Theta\dot{\eta}] + \mathbf{T} \cdot \nabla\dot{\mathbf{u}} - \operatorname{div} \mathbf{q} = 0, \tag{3.3}$$

where $r(\mathbf{x}, t)$ is the specific heat supply field, $\eta(\mathbf{x}, t)$ is the specific entropy field, $\mathbf{q}(\mathbf{x}, t)$ is the heat flux vector, and $\mathbf{T}(\mathbf{x}, t) = \mathbf{T}^T(\mathbf{x}, t)$ is the stress tensor field.

The local form of the Clausius–Duhem inequality reads

$$\rho\Theta\dot{\eta} - \rho r + \operatorname{div} \mathbf{q} - \frac{\mathbf{q} \cdot \nabla\theta}{\Theta} \geqslant 0 \tag{3.4}$$

and the elimination of r between (3.3) and (3.4) yields

$$-\rho\dot{\Psi} - \rho\dot{\theta}\eta + \mathbf{T} \cdot \nabla\dot{\mathbf{u}} - \frac{\mathbf{q} \cdot \nabla\theta}{\theta_0} \geqslant 0, \tag{3.5}$$

where we have retained only terms of first order in $\nabla\theta/\Theta$.

The local entropy production inequality (3.5) must be satisfied for any process undergone by the body. Substituting for $\rho\dot{\Psi}$ from (3.1) into (3.5), rearranging terms, and taking into account that the resulting inequality must hold for arbitrary values of $\nabla\dot{\mathbf{u}}(t)$ and $\dot{\theta}(t)$, it follows that[4]

$$\mathbf{T}(t) = \mathbf{G}(0)\,\nabla\mathbf{u}(t) - \theta(t)\,\mathbf{L}(0) + \int_{-\infty}^{t} \dot{\mathbf{G}}(t-\tau)\,\nabla\mathbf{u}(\tau)\,d\tau - \int_{-\infty}^{t} \dot{\mathbf{L}}(t-\tau)\,\theta(\tau)\,d\tau, \tag{3.6}$$

$$\begin{aligned}\rho\eta(t) = \mathbf{L}(0) \cdot \nabla\mathbf{u}(t) + \rho\,\frac{c(0)}{\theta_0}\,\theta(t) + \int_{-\infty}^{t} \dot{\mathbf{L}}(t-\tau) \cdot \nabla\mathbf{u}(\tau)\,d\tau \\ + \frac{\rho}{\theta_0}\int_{-\infty}^{t} \dot{c}(t-\tau)\,\theta(\tau)\,d\tau,\end{aligned} \tag{3.7}$$

[3] Within the general framework of the theory of thermodynamics of simple materials with fading memory, Coleman [17] has studied the consequences of the Clausius–Duhem inequality on the stress relaxation function of linear viscoelasticity in isothermal conditions. He concludes that $\mathbf{G}(0)$, $\mathbf{G}(\infty)$ are both symmetric and their difference, $\mathbf{G}(0) - \mathbf{G}(\infty)$, is positive semidefinite. A stronger assumption, the dissipativity requirement, implies furthermore (Gurtin and Herrera [18]) that $\mathbf{G}(\infty)$ is positive semidefinite. However, the symmetry of $\mathbf{G}(s)$ for every s is not a consequence of either the Clausius–Duhem inequality or the dissipativity requirement, as is shown by Shu and Onat [19] and Day [20] by constructing a counterexample. The latter author [21] gives a different condition which is equivalent to $(3.2)_1$.

[4] We assume that $\mathbf{T}$ and η are functionals depending upon the histories of $\nabla\mathbf{u}$ and θ.

which are the constitutive equations for the stress tensor and specific entropy difference.

We complete the set of constitutive equations by restricting our attention to Fourier's law for the heat flux vector:

$$-\mathbf{q}(\mathbf{x}, t) = \mathbf{K}(\mathbf{x})\,\nabla\theta(t), \tag{3.8}$$

where $\mathbf{K}(\mathbf{x})$ is the thermal conductivity tensor field.

After (3.6), (3.7), the inequality (3.5) reduces to the inequality

$$\rho\delta - \frac{\mathbf{q}\cdot\nabla\theta}{\theta_0} \geqslant 0 \tag{3.9}$$

where δ, the internal dissipation, is given by

$$\begin{aligned}\rho\delta(t) = \tfrac{1}{2}\int_{-\infty}^{t} [\nabla\mathbf{u}(t) - \nabla\mathbf{u}(\tau)]\cdot\ddot{\mathbf{G}}(t-\tau)\,[\nabla\mathbf{u}(t) - \nabla\mathbf{u}(\tau)]\,d\tau \\ -\frac{\rho}{2\theta_0}\int_{-\infty}^{t} \ddot{c}(t-\tau)\,[\theta(t) - \theta(\tau)]^2\,d\tau \\ -\int_{-\infty}^{t} [\theta(t) - \theta(\tau)]\,\ddot{\mathbf{L}}(t-\tau)\cdot[\mathbf{u}(t) - \nabla\mathbf{u}(\tau)]\,d\tau.\end{aligned} \tag{3.10}$$

By considering a particular process with $\nabla\theta = \mathbf{0}$ for which (3.9) must apply, we have

$$\delta \geqslant 0. \tag{3.11}$$

Therefore, if

$$-\ddot{c}(s) = c_2(s) \geqslant 0, \qquad s \geqslant 0 \tag{3.12}$$

and

$$\mathbf{M}\cdot\ddot{\mathbf{G}}(s)\,\mathbf{M} \geqslant g_2(s)\,|\,\mathbf{M}\,|^2, \qquad g_2(s) \geqslant 0, \quad s \geqslant 0 \tag{3.13}$$

for every second-order tensor $\mathbf{M}$, and furthermore,

$$|\,\ddot{\mathbf{L}}(s)| \leqslant (\rho/\theta_0)^{1/2}\,\{g_2(s)\}^{1/2}\,\{c_2(s)\}^{1/2}, \tag{3.14}$$

then the inequality (3.11) is obviously satisfied.

Using (3.11) a sufficient condition for (3.9) to be met is found to be $\mathbf{q}\cdot\nabla\theta \leqslant 0$, which implies, in view of (3.8), the positive semidefiniteness of the thermal conductivity tensor $\mathbf{K}$.

The local form (3.3) of the balance of energy law can be rewritten as

$$\rho r + \rho\delta - \rho\Theta\dot{\eta} - \operatorname{div}\mathbf{q} = 0.$$

But δ is a second-order term and $\Theta = \theta_0 +$ first-order term; hence, for a consistent first-order theory, we must have

$$\rho r - \rho\theta_0\dot{\eta} - \operatorname{div}\mathbf{q} = 0 \tag{3.15}$$

as the linearized version of the energy equation. The remaining field equation is the local form of balance of linear momentum:

$$\operatorname{div}\mathbf{T} + \rho\mathbf{b} = \rho\ddot{\mathbf{u}}, \tag{3.16}$$

where $\mathbf{b}$ is the specific body force.

Substituting (3.6), (3.7), (3.8) in (3.15), (3.16), and assuming that the required smoothness properties are met, we obtain

$$\begin{aligned}\rho\ddot{\mathbf{u}}(t) = \operatorname{div}\Big[&\mathbf{G}(0)\,\nabla\mathbf{u}(t) - \theta(t)\,\mathbf{L}(0) + \int_{-\infty}^{t}\dot{\mathbf{G}}(t-\tau)\,\nabla\mathbf{u}(\tau)\,d\tau\\ &- \int_{-\infty}^{t}\dot{\mathbf{L}}(t-\tau)\,\theta(\tau)\,d\tau\Big] + \rho\mathbf{b}(t),\end{aligned} \tag{3.17}$$

$$\begin{aligned}\operatorname{div}[\mathbf{K}\nabla\theta(t)] = \theta_0\frac{\partial}{\partial t}\Big[&\mathbf{L}(0)\cdot\nabla\mathbf{u}(t) + \int_{-\infty}^{t}\dot{\mathbf{L}}(t-\tau)\cdot\nabla\mathbf{u}(\tau)\,d\tau\\ &+ \rho\frac{c(0)}{\theta_0}\,\theta(t) + \frac{\rho}{\theta_0}\int_{-\infty}^{t}\dot{c}(t-\tau)\,\theta(\tau)\,d\tau\Big] - \rho r(t)\end{aligned} \tag{3.18}$$

as the system formed by the equation of motion and the coupled heat conduction equation for the linear thermoviscoelasticity theory.

We now formulate our boundary-initial value problem in a classical sense. Let the body B occupy a bounded regular domain of R^3 with boundary ∂B, that is, a bounded domain such that the Green–Gauss formula is applicable [22].[5]

We assume that the externally applied body forces $\mathbf{b}$ vanish identically, except for possibly their application in $t < 0$, that is,

$$\mathbf{b}(\mathbf{x}, t) = \mathbf{0} \qquad \text{on} \qquad B \times [0, \infty). \tag{3.19}$$

In including long-distance thermal effects, we assume that the external heat supply r is identically zero for $t \geqslant 0$:

$$r(\mathbf{x}, t) = 0 \qquad \text{on} \qquad B \times [0, \infty). \tag{3.20}$$

We also assume that displacement and temperature difference fields are known up to time $t = 0$, namely,

$$\begin{aligned}\mathbf{u}(\mathbf{x}, t) &= \mathbf{u}_1(\mathbf{x}, t) \qquad \text{on} \qquad \bar{B} \times (-\infty, 0],\\ \theta(\mathbf{x}, t) &= \theta_1(\mathbf{x}, t) \qquad \text{on} \qquad \bar{B} \times (-\infty, 0],\end{aligned} \tag{3.21}$$

[5] By bounded domain we mean an open, bounded, and connected set.

where $\mathbf{u}_1(\mathbf{x}, t)$, $\theta_1(\mathbf{x}, t)$ are given functions. Hence, using (3.19), (3.20), (3.21), we can cast (3.17), (3.18) into the form

$$\rho\ddot{\mathbf{u}}(t) = \operatorname{div}\Big[\mathbf{G}(0)\,\nabla\mathbf{u}(t) - \theta(t)\,\mathbf{L}(0) + \int_0^t \dot{\mathbf{G}}(t-\tau)\,\nabla\mathbf{u}(\tau)\,d\tau - \int_0^t \dot{\mathbf{L}}(t-\tau)\,\theta(\tau)\,d\tau\Big] + \mathbf{b}_0(t)$$

$$\operatorname{div}[\mathbf{K}\nabla\theta(t)] + r_0(t) = \theta_0\frac{\partial}{\partial t}\Big[\mathbf{L}(0)\cdot\nabla\mathbf{u}(t) + \rho\frac{c(0)}{\theta_0}\theta(t) + \int_0^t \dot{\mathbf{L}}(t-\tau)\cdot\nabla\mathbf{u}(\tau)\,d\tau + \frac{\rho}{\theta_0}\int_0^t \dot{c}(t-\tau)\,\theta(\tau)\,d\tau\Big], \tag{3.22}$$

where

$$\begin{aligned}\mathbf{b}_0(t) &\equiv \operatorname{div}\Big[\int_{-\infty}^0 \dot{\mathbf{G}}(t-\tau)\,\nabla\mathbf{u}_1(\tau)\,d\tau - \int_{-\infty}^0 \dot{\mathbf{L}}(t-\tau)\,\theta_1(\tau)\,d\tau\Big],\\ r_0(t) &\equiv -\theta_0\frac{\partial}{\partial t}\Big[\int_{-\infty}^0 \dot{\mathbf{L}}(t-\tau)\cdot\nabla\mathbf{u}_1(\tau)\,d\tau + \frac{\rho}{\theta_0}\int_{-\infty}^0 \dot{c}(t-\tau)\,\theta_1(\tau)\,d\tau\Big]\end{aligned} \tag{3.23}$$

can be regarded as assigned body force term and heat supply term, respectively, both arising from the past history of strain and temperature.

We assume the homogeneous boundary condition of place

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{0} \qquad \text{on} \qquad \partial B \times [0, \infty) \tag{3.24}$$

together with the thermal condition

$$\theta(\mathbf{x}, t) = 0 \qquad \text{on} \qquad \partial B \times [0, \infty) \tag{3.25}$$

which states that the boundary is maintained at constant temperature θ_0.

Let a fixed time interval $(0, t_0)$ be given. A classical solution of the initial history-boundary value problem in $B \times (0, t_0)$ is a pair $(\mathbf{u}, \theta)$ satisfying (3.17), (3.18), with (3.19), (3.20), on $B \times (0, t_0)$, together with (3.21) and (3.24), (3.25) on $\partial B \times [0, t_0)$. Alternatively, a classical solution can be considered as a pair $(\mathbf{u}, \theta)$ satisfying (3.22) on $B \times (0, t_0)$, together with (3.24), (3.25) on $\partial B \times [0, t_0)$ and the initial conditions

$$(\mathbf{u}_I(\mathbf{x}), \dot{\mathbf{u}}_I(\mathbf{x}), \theta_I(\mathbf{x})) = (\mathbf{u}_1(\mathbf{x}, 0), \dot{\mathbf{u}}_1(\mathbf{x}, 0), \theta_1(\mathbf{x}, 0)) \qquad \text{on} \qquad \bar{B}. \tag{3.26}$$

We turn now to deriving an identity satisfied for any classical solution. Let us consider the set of function pairs

$$\{(\mathbf{w}, \beta) \mid \mathbf{w} \in C^\infty([0, t_0]; \boldsymbol{C}_0^1(B)), \mathbf{w}(t_0) = \mathbf{0} \text{ on } B \text{ and } \beta \in C^\infty([0, t_0]; C_0^1(B)), \beta(t_0) = 0 \text{ on } B\}.$$

We form the scalar product of $(3.22)_1$, $(3.22)_2$, and $\mathbf{w}(t)$, $\beta(t)$, respectively, and we integrate over $B \times (0, t_0)$ and sum. Performing integration by parts and using (3.26), together with the particular choice $\mathbf{w}(t) = (t - t_0)\,\dot{\mathbf{v}}(t)$ and $\beta(t) = (t - t_0)\,\alpha(t)/\theta_0$, we have that any classical solution $(\mathbf{u}, \theta)$ will satisfy

$$A[(\mathbf{u}, \theta), (\mathbf{v}, \alpha)] = I[(\mathbf{u}_I, \dot{\mathbf{u}}_I, \theta_I), (\mathbf{v}, \alpha)] + S[(\mathbf{b}_0, r_0), (\mathbf{v}, \alpha)] \tag{3.27}$$

for all $(\mathbf{v}, \alpha) \in C^\infty([0, t_0]; C_0^1(B)) \times C^\infty([0, t_0]; C_0^1(B))$, where

$$\begin{aligned}
&A[(\mathbf{u}, \theta), (\mathbf{v}, \alpha)] \\
&= \int_0^{t_0}\int_B \Big\{(t - t_0)\Big[\rho\dot{\mathbf{u}}(t)\cdot\ddot{\mathbf{v}}(t) - \nabla\dot{\mathbf{v}}(t)\cdot\mathbf{G}(0)\,\nabla\mathbf{u}(t) + \theta(t)\,\mathbf{L}(0)\cdot\nabla\dot{\mathbf{v}}(t) \\
&\quad - \int_0^t \nabla\dot{\mathbf{v}}(t)\cdot\dot{\mathbf{G}}(t-\tau)\,\nabla\mathbf{u}(\tau)\,d\tau + \int_0^t \theta(\tau)\,\dot{\mathbf{L}}(t-\tau)\cdot\nabla\dot{\mathbf{v}}(t)\,d\tau \\
&\quad + \dot{\alpha}(t)\,\mathbf{L}(0)\cdot\nabla\mathbf{u}(t) + \int_0^t \dot{\alpha}(t)\,\dot{\mathbf{L}}(t-\tau)\cdot\nabla\mathbf{u}(\tau)\,d\tau \\
&\quad + \rho\,\frac{c(0)}{\theta_0}\,\dot{\alpha}(t)\,\theta(t) + \frac{\rho}{\theta_0}\int_0^t \dot{\alpha}(t)\,\dot{c}(t-\tau)\,\theta(\tau)\,d\tau\Big] \\
&\quad + \rho\dot{\mathbf{u}}(t)\cdot\dot{\mathbf{v}}(t) + \alpha(t)\,\mathbf{L}(0)\cdot\nabla\mathbf{u}(t) + \int_0^t \alpha(t)\,\dot{\mathbf{L}}(t-\tau)\cdot\nabla\mathbf{u}(\tau)\,d\tau \\
&\quad + \rho\,\frac{c(0)}{\theta_0}\,\alpha(t)\,\theta(t) + \frac{\rho}{\theta_0}\int_0^t \alpha(t)\,\dot{c}(t-\tau)\,\theta(\tau)\,d\tau \\
&\quad + 1/\theta_0\int_0^t \nabla\alpha(\tau)\cdot\mathbf{K}\nabla\theta(\tau)\,d\tau\Big\}\,d\mathbf{x}\,dt,
\end{aligned}$$

$$\begin{aligned}
&I[(\mathbf{u}_I, \dot{\mathbf{u}}_I, \theta_I), (\mathbf{v}, \alpha)] \\
&= t_0\int_B \Big[\rho\dot{\mathbf{v}}(0)\cdot\dot{\mathbf{u}}_I + \alpha(0)\,\mathbf{L}(0)\cdot\nabla\mathbf{u}_I + \rho\,\frac{c(0)}{\theta_0}\,\alpha(0)\,\theta_I\Big]\,d\mathbf{x},
\end{aligned} \tag{3.29}$$

and

$$\begin{aligned}
&S[(\mathbf{b}_0, r_0), (\mathbf{v}, \alpha)] \\
&= -\int_0^{t_0}\int_B (t - t_0)\Big[\dot{\mathbf{v}}(t)\cdot\mathbf{b}_0(t) + \frac{1}{\theta_0}\,\alpha(t)\,r_0(t)\Big]\,d\mathbf{x}\,dt.
\end{aligned} \tag{3.30}$$

Through (3.27) we define in the next section a generalized solution to the problem defined by the Eqs. (3.22), (3.24), (3.25), (3.26) and prove its existence and uniqueness. That solution will not be in general a classical solution, unless additional hypothesis are made.

4. Properties of Solutions

After having proved in the present section the existence and uniqueness of a generalized solution, we investigate its smoothness properties. For that purpose we introduce spaces endowed with norms of the fading memory type studied by Coleman and Mizel [12].

Let the region occupied by the body B be a bounded domain of R^3. We assume that for fixed $s \in [0, \infty)$ the material properties $\mathbf{G}(\mathbf{x}, s)$, $\mathbf{L}(\mathbf{x}, s)$, $c(\mathbf{x}, s)$, $\mathbf{K}(\mathbf{x})$ are Lebesgue measurable functions, essentially bounded on B, and with norms

$$\| \overset{(i)}{\mathbf{G}}(s) \| = \operatorname*{ess.\,sup}_{\mathbf{x} \in B} | \overset{(i)}{\mathbf{G}}(\mathbf{x}, s) | , \qquad \| \overset{(i)}{\mathbf{L}}(s) \| = \operatorname*{ess.\,sup}_{\mathbf{x} \in B} | \overset{(i)}{\mathbf{L}}(\mathbf{x}, s) | ,$$

$$\| \overset{(i)}{c}(s) \| = \operatorname*{ess.\,sup}_{\mathbf{x} \in B} | \overset{(i)}{c}(\mathbf{x}, s) | ,$$

where $i = 0, 1, 2$. It is also assumed that $\mathbf{L}(\mathbf{x}, s)$ satisfies $(3.2)_2$, i.e.,

$$\mathbf{L}(\mathbf{x}, s) = \mathbf{L}^T(\mathbf{x}, s), \qquad s \geqslant 0 \tag{4.1}$$

and that[6]

$$\mathbf{G}(\mathbf{x}, 0) = \mathbf{G}^T(\mathbf{x}, 0) \tag{4.2}$$

almost everywhere on B.

We make the following further assumptions:

(a) $$0 < \rho_0 \leqslant \operatorname*{ess.\,inf}_{\mathbf{x} \in B} \rho(\mathbf{x}) \leqslant \operatorname*{ess.\,sup}_{\mathbf{x} \in B} \rho(\mathbf{x}) \leqslant \rho_1 . \tag{4.3}$$

(b) $$\operatorname*{ess.\,inf}_{\mathbf{x} \in B} c(\mathbf{x}, 0) \geqslant c_0 , \qquad c_0 > 0. \tag{4.4}$$

(c) There exists a positive constant g_0 such that

$$\int_B \nabla \mathbf{v} \cdot \mathbf{G}(0) \nabla \mathbf{v} \, d\mathbf{x} \geqslant g_0 \| \mathbf{v} \|_1^2 \qquad \text{for all } \mathbf{v} \in \mathring{W}_2^1(B). \tag{4.5}$$

(d) There exists a positive constant K such that

$$\int_B \nabla \alpha \cdot \mathbf{K} \nabla \alpha \, d\mathbf{x} \geqslant K \| \alpha \|_1^2 \qquad \text{for all } \alpha \in \mathring{W}_2^1(B). \tag{4.6}$$

[6] We assume in this section that the material is defined from the outset by the constitutive equations (3.6), (3.7), (3.8). Therefore, the symmetry of $\mathbf{G}(s)$ for all $s > 0$ is not assumed and is not needed in the subsequent theorems concerning the mixed initial-boundary value problem for the evolution equations (3.22).

(e) $\|\dot{\mathbf{G}}(s)\|$, $\|\ddot{\mathbf{G}}(s)\|$, $\|\dot{c}(s)\|$, $\|\ddot{c}(s)\|$ are continuous functions on $[0, \infty)$. Furthermore,

$$\|\overset{(i)}{\mathbf{G}}(s)\| \leqslant Ri(s), \qquad \|\overset{(i)}{c}(s)\| \leqslant Ri(s), \qquad i = 1, 2, \tag{4.7}$$

where R is a constant, $i(s) \in L_1(0, \infty)$, and $i(s) > 0$ almost everywhere. Moreover, the function $i(s)$ satisfies the condition that the values of

$$\operatorname*{ess.\,sup}_{s\in(0,\infty)} i(s+t)/i(s) \tag{4.8}$$

are finite for all $t \in [0, \infty)$.[7]

(f) For all $s \geqslant 0$

$$\|\dot{\mathbf{L}}(s)\| \leqslant (\rho_1/\theta_0)^{1/2}\,(\|\dot{\mathbf{G}}(s)\|)^{1/2}\,(\|\dot{c}(s)\|)^{1/2}, \tag{4.9}$$

$$\|\ddot{\mathbf{L}}(s)\| \leqslant (\rho_1/\theta_0)^{1/2}\,(\|\ddot{\mathbf{G}}(s)\|)^{1/2}\,(\|\ddot{c}(s)\|)^{1/2}. \tag{4.10}$$

Remark 4.1. Obviously, condition (4.10) finds a motivation in the inequality (3.14). As we shall show in the next section, conditions (3.12), (3.13), (3.14), together with the facts $\lim_{s\to\infty} \|\dot{\mathbf{G}}(s)\| = \lim_{s\to\infty} \|\dot{c}(s)\| = 0$, implied by (4.7), and $\lim_{s\to\infty} \|\dot{\mathbf{L}}(s)\| = 0$, lead to a stronger condition than inequality (4.9) as well as some monotonicity restrictions upon $-\dot{\mathbf{G}}(s)$, $\dot{c}(s)$. Assumptions (4.3), (4.4) are physically natural. We have already stated in the footnote 3 how (4.5), (4.2) may be interpreted. Condition (4.6) is clearly related to the heat conduction inequality $\mathbf{q} \cdot \nabla\theta \leqslant 0$.

DEFINITION 4.1. Choose any $t_0 > 0$. We denote by P_{t_0} the Hilbert space obtained as the completion of the set

$$\{(\mathbf{v}, \alpha) \mid (\mathbf{v}, \alpha) \in C^\infty([0, t_0];\ \mathring{W}_2{}^1(B)) \times C^\infty([0, t_0];\ \mathring{W}_2{}^1(B))\}$$

under the norm $\|\cdot\|$ induced by the inner product

$$\langle(\mathbf{v}_1, \alpha_1), (\mathbf{v}_2, \alpha_2)\rangle_{P_{t_0}}$$
$$= \int_0^{t_0}\int_B \left[\dot{\mathbf{v}}_1(t)\cdot\dot{\mathbf{v}}_2(t) + \nabla\mathbf{v}_1(t)\cdot\nabla\mathbf{v}_2(t) + \alpha_1(t)\,\alpha_2(t)\int_0^t \nabla\alpha_1(\tau)\cdot\nabla\alpha_2(\tau)\,d\tau\right] d\mathbf{x}\,dt. \tag{4.11}$$

[7] The function $i(s)$ is an "influence function" of the type used in [12]. If $i(s)$ is monotone decreasing on $(0, \infty)$, (4.8) will be bounded on $[0, \infty)$ and the relaxation property will hold. If, in addition, $i(s)$ is such that

$$\lim_{t\to\infty} \operatorname*{ess.\,sup}_{s\in(0,\infty)} i(s+t)/i(s) = 0 \tag{4.12}$$

then $i(s)$ satisfies $a_1 e^{-b_1 s} < i(s) < a_2 e^{-b_2 s}$ a.e. on $(0, \infty)$ for some positive constants a_1, a_2, b_1, b_2.

DEFINITION 4.2. By T_{t_0} we denote the set

$$\{(\mathbf{v}, \alpha) \mid \mathbf{v} \in C^{\infty}([0, t_0]; \mathring{\boldsymbol{W}}_2^1(B)), \mathbf{v}(0) = \mathbf{0} \text{ on } B \text{ and } \alpha \in C^{\infty}([0, t_0]; \mathring{W}_2^1(B))\}.$$

We equip the set T_{t_0} with the inner product

$$\begin{aligned}
&\langle(\mathbf{v}_1, \alpha_1), (\mathbf{v}_2, \alpha_2)\rangle_{T_{t_0}} \\
&= \int_0^{t_0}\int_B \left[\dot{\mathbf{v}}_1(t) \cdot \dot{\mathbf{v}}_2(t) + \nabla\mathbf{v}_1(t) \cdot \nabla\mathbf{v}_2(t) + \alpha_1(t)\,\alpha_2(t) + \int_0^t \nabla\alpha_1(\tau) \cdot \nabla\alpha_2(\tau)\,d\tau\right] d\mathbf{x}\,dt \\
&\quad + t_0 \int_B [\dot{\mathbf{v}}_1(0) \cdot \dot{\mathbf{v}}_2(0) + \alpha_1(0)\,\alpha_2(0)]\,d\mathbf{x} \qquad (4.13)
\end{aligned}$$

which induces the norm $|||\cdot|||$.

DEFINITION 4.3. By $\mathbf{H}(B)$ we denote the space completion of the set $\{\mathbf{b} \mid \mathbf{b} \in \mathbf{L}_2(B)\}$ under the norm

$$\|\mathbf{b}\|_{-1} = \sup_{\mathbf{v} \in \mathring{\boldsymbol{W}}_2^1(B)} \left|\int_B \mathbf{b} \cdot \mathbf{v}\,d\mathbf{x}\right| \Big/ \|\mathbf{v}\|_1 . \qquad (4.14)$$

Let $t_0 > 0$ and let A, I, S be defined as in (3.28), (3.29), (3.30), respectively. It is easy to see that all these definitions make sense for the more general spaces considered here. Therefore, we may state:

DEFINITION 4.4. A generalized solution of the equations (3.22), (3.24), (3.25) in $B \times [0, t_0]$, with initial conditions $(\mathbf{u}_I, \dot{\mathbf{u}}_I, \theta_I) \in \mathring{\boldsymbol{W}}_2^1(B) \times \boldsymbol{L}_2(B) \times L_2(B)$ and supply term $(\mathbf{b}_0, r_0) \in L_2([0, t_0]; \boldsymbol{H}(B) \times L_2(B))$ is a pair $(\mathbf{u}, \theta) \in P_{t_0}$ satisfying

$$A[(\mathbf{u}, \theta), (\mathbf{v}, \alpha)] = I[(\mathbf{u}_I, \dot{\mathbf{u}}_I, \theta_I), (\mathbf{v}, \alpha)] + S[(\mathbf{b}_0, r_0), (\mathbf{v}, \alpha)] \qquad (4.15)$$

for all $(\mathbf{v}, \alpha)$ in the set T_{t_0}.

This definition of a generalized solution is meaningful since by (3.27) a solution in the classical sense is a generalized solution. Also, if $(\mathbf{u}, \theta)$ is a generalized solution which is sufficiently smooth, then it is a classical solution.

LEMMA 4.1. *If $t_1 > 0$ is sufficiently small, there exists a positive constant C_A such that*

$$A[(\mathbf{v}, \alpha), (\mathbf{v}, \alpha)] \geqslant C_A |||(\mathbf{v}, \alpha)|||^2 \qquad (4.16)$$

for all $(\mathbf{v}, \alpha) \in T_{t_1}$. C_A and t_1 depend only on the material.

Proof. Integrating by parts the expression for $A[(\mathbf{v}, \alpha), (\mathbf{v}, \alpha)]$, and using (4.2) together with $\mathbf{v}(0) = \mathbf{0}$ on B, we obtain

$$
\begin{aligned}
&A[(\mathbf{v}, \alpha), (\mathbf{v}, \alpha)] \\
&= \int_0^{t_1}\int_B \left\{\frac{1}{2}\rho\dot{\mathbf{v}}(t)\cdot\dot{\mathbf{v}}(t) + \frac{1}{2}\nabla\mathbf{v}(t)\cdot\mathbf{G}(0)\,\nabla\mathbf{v}(t) + \frac{1}{2}\rho\frac{c(0)}{\theta_0}\alpha(t)^2\right\} d\mathbf{x}\,dt \\
&\quad + t_1\int_B \left\{\frac{1}{2}\rho\dot{\mathbf{v}}(0)\cdot\dot{\mathbf{v}}(0) + \frac{1}{2}\rho\frac{c(0)}{\theta_0}\alpha(0)^2\right\} d\mathbf{x} \\
&\quad + \frac{1}{\theta_0}\int_0^{t_1}\int_B\int_0^t \nabla\alpha(\tau)\cdot\mathbf{K}\nabla\alpha(\tau)\,d\tau\,d\mathbf{x}\,dt \\
&\quad - \Big[-\int_0^{t_1}\int_B (t-t_1)\Big\{\nabla\mathbf{v}(t)\cdot\dot{\mathbf{G}}(0)\,\nabla\mathbf{v}(t) - \frac{\rho}{\theta_0}\dot{c}(0)\,\alpha(t)^2 \\
&\quad - 2\alpha(t)\,\dot{\mathbf{L}}(0)\cdot\nabla\mathbf{v}(t)\Big\}\,d\mathbf{x}\,dt \\
&\quad + \int_0^{t_1}\int_B (t-t_1)\Big\{-\int_0^t \nabla\mathbf{v}(t)\cdot\ddot{\mathbf{G}}(t-\tau)\,\nabla\mathbf{v}(\tau)\,d\tau + \frac{\rho}{\theta_0}\int_0^t \ddot{c}(t-\tau)\,\alpha(t)\,\alpha(\tau)\,d\tau \\
&\quad + \int_0^t \alpha(t)\,\ddot{\mathbf{L}}(t-\tau)\cdot\nabla\mathbf{v}(\tau)\,d\tau + \int_0^t \alpha(\tau)\,\ddot{\mathbf{L}}(t-\tau)\cdot\nabla\mathbf{v}(t)\,d\tau\Big\}\,d\mathbf{x}\,dt \\
&\quad + \int_0^{t_1}\int_B \Big\{-\int_0^t \nabla\mathbf{v}(t)\cdot\dot{\mathbf{G}}(t-\tau)\,\nabla\mathbf{v}(\tau)\,d\tau + \int_0^t \alpha(\tau)\,\dot{\mathbf{L}}(t-\tau)\cdot\nabla\mathbf{v}(t)\,d\tau\Big\}\,d\mathbf{x}\,dt\Big]
\end{aligned}
$$

Using the Cauchy–Schwarz inequality to estimate the middle four terms in the square bracket, we have

$$
\begin{aligned}
&\Big|\int_0^{t_1}\int_B (t-t_1)\Big\{-\int_0^t \nabla\mathbf{v}(t)\cdot\ddot{\mathbf{G}}(t-\tau)\,\nabla\mathbf{v}(\tau)\,d\tau + \frac{\rho}{\theta_0}\int_0^t \ddot{c}(t-\tau)\,\alpha(t)\,\alpha(\tau)\,d\tau \\
&\qquad + \int_0^t \alpha(t)\,\ddot{\mathbf{L}}(t-\tau)\cdot\nabla\mathbf{v}(\tau)\,d\tau + \int_0^t \alpha(\tau)\,\ddot{\mathbf{L}}(t-\tau)\cdot\nabla\mathbf{v}(t)\,d\tau\Big\}\,d\mathbf{x}\,dt\Big| \\
&\leqslant t_1\Big\{\sup_{s\in[0,t_1]}\|\ddot{\mathbf{G}}(s)\|\Big(\int_0^{t_1}\|\mathbf{v}(t)\|_1\,dt\Big)^2 + \frac{\rho_1}{\theta_0}\sup_{s\in[0,t_1]}\|\ddot{c}(s)\|\Big(\int_0^{t_1}\|\alpha(t)\|_0\,dt\Big)^2 \\
&\qquad + 2\operatorname*{ess.\,sup}_{s\in[0,t_1]}\|\ddot{\mathbf{L}}(s)\|\Big(\int_0^{t_1}\|\mathbf{v}(t)\|_1\,dt\Big)\Big(\int_0^{t_1}\|\alpha(t)\|_0\,dt\Big)\Big\} \\
&\leqslant t_1\Big\{\sup_{[0,t_1]}\|\ddot{\mathbf{G}}(s)\|\Big(\int_0^{t_1}\|\mathbf{v}(t)\|_1\,dt\Big)^2 + \frac{\rho_1}{\theta_0}\sup_{[0,t_1]}\|\ddot{c}(s)\|\Big(\int_0^{t_1}\|\alpha(t)\|_0\,dt\Big)^2
\end{aligned}
$$

$$+ 2(\rho_1/\theta_0)^{1/2} (\sup_{[0,t_1]} \| \ddot{\mathbf{G}}(s)\|)^{1/2} (\sup_{[0,t_1]} \| \ddot{c}(s)\|)^{1/2} \left(\int_0^{t_1} \| \mathbf{v}(t)\|_1 \, dt\right) \left(\int_0^{t_1} \| \alpha(t)\|_0 dt\right)\Big\}$$

$$\leqslant 2t_1 \Big\{\sup_{[0,t_1]} \| \ddot{\mathbf{G}}(s)\| \left(\int_0^{t_1} \| \mathbf{v}(t)\|_1 \, dt\right)^2 + \frac{\rho_1}{\theta_0} \sup_{[0,t_1]} \| \ddot{c}(s)\| \left(\int_0^{t_1} \| \alpha(t)\|_0 \, dt\right)^2\Big\}$$

$$\leqslant 2t_1{}^2 \Big\{\sup_{[0,t_1]} \| \ddot{\mathbf{G}}(s)\| \int_0^{t_1} \| \mathbf{v}(t)\|_1^2 \, dt + \frac{\rho_1}{\theta_0} \sup_{[0,t_1]} \| \ddot{c}(s)\| \int_0^{t_1} \| \alpha(t)\|_0^2 \, dt\Big\},$$

where we have made use of (4.10). Similarly, applying the Cauchy–Schwarz inequality and using (4.9), we obtain

$$\left| \int_0^t \int_B \Big\{- \int_0^t \nabla \mathbf{v}(t) \cdot \dot{\mathbf{G}}(t-\tau) \nabla \mathbf{v}(\tau) \, d\tau + \int_0^t \alpha(\tau) \dot{\mathbf{L}}(t-\tau) \cdot \nabla \mathbf{v}(t) \, d\tau\Big\} \, d\mathbf{x} \, dt \right|$$

$$\leqslant 2t_1 \Big\{\sup_{[0,t_1]} \| \dot{\mathbf{G}}(s)\| \int_0^{t_1} \| \mathbf{v}(t)\|_1^2 \, dt + \frac{\rho_1}{\theta_0} \sup_{[0,t_1]} \| \dot{c}(s)\| \int_0^{t_1} \| \alpha(t)\|_0^2 \, dt\Big\}.$$

Finally, and in the same way,

$$\left| \int_0^{t_1} \int_B (t - t_1) \Big\{-\nabla \mathbf{v}(t) \cdot \dot{\mathbf{G}}(0) \nabla \mathbf{v}(t) + \frac{\rho}{\theta_0} \dot{c}(0) \, \alpha(t)^2 + 2\alpha(t) \dot{\mathbf{L}}(0) \cdot \nabla \mathbf{v}(t)\Big\} \, d\mathbf{x} \, dt \right|$$

$$\leqslant 2t_1 \Big\{\| \dot{\mathbf{G}}(0)\| \int_0^{t_1} \| \mathbf{v}(t)\|_1^2 \, dt + \frac{\rho_1}{\theta_0} \| \dot{c}(0)\| \int_0^{t_1} \| \alpha(t)\|_0^2 \, dt\Big\}.$$

Therefore, using the estimates above, together with (4.3), (4.4), (4.5), (4.6), we obtain

$$A[(\mathbf{v}, \alpha), (\mathbf{v}, \alpha)]$$

$$\geqslant \frac{\rho_0}{2} \int_0^{t_1} \| \dot{\mathbf{v}}(t)\|_0^2 \, dt + \frac{g_0}{2} \int_0^{t_1} \| \mathbf{v}(t)\|_1^2 \, dt + \frac{\rho_0 c_0}{2\theta_0} \int_0^{t_1} \| \alpha(t)\|_0^2 \, dt$$

$$+ \frac{t_1 \rho_0}{2} \| \dot{\mathbf{v}}(0)\|_0^2 + \frac{t_1 \rho_0 c_0}{2\theta_0} \| \alpha(0)\|_0^2 + \frac{K}{\theta_0} \int_0^{t_1} \int_0^t \| \alpha(\tau)\|_1^2 \, d\tau \, dt$$

$$- \Big\{t_1(2t_1 \sup_{[0,t_1]} \| \ddot{\mathbf{G}}(s)\| + 2 \sup_{[0,t_1]} \| \dot{\mathbf{G}}(s)\| + 2 \| \dot{\mathbf{G}}(0)\|) \int_0^{t_1} \| \mathbf{v}(t)\|_1^2 \, dt$$

$$+ t_1 \left(\frac{2t_1 \rho_1}{\theta_0} \sup_{[0,t_1]} \| \ddot{c}(s)\| + \frac{2\rho_1}{\theta_0} \sup_{[0,t_1]} \| \dot{c}(s)\| + \frac{2\rho_1}{\theta_0} \| \dot{c}(0)\|\right) \int_0^{t_1} \| \alpha(t)\|_0^2 \, dt\Big\}.$$

If we choose t_1 small enough, we can get

$$t_1(2t_1 \sup_{[0,t_1]} \| \ddot{\mathbf{G}}(s)\| + 2 \sup_{[0,t_1]} \| \dot{\mathbf{G}}(s)\| + 2 \| \dot{\mathbf{G}}(0)\|) \leqslant \frac{g_0}{4}$$

and

$$t_1 \left(\frac{2t_1\rho_1}{\theta_0} \sup_{[0,t_1]} \| \ddot{c}(s)\| + \frac{2\rho_1}{\theta_0} \sup_{[0,t_1]} \| \dot{c}(s)\| + \frac{2\rho_1}{\theta_0} \| \dot{c}(0)\| \right) \leqslant \frac{\rho_0 c_0}{4\theta_0}$$

and thus, t_1 turns out to depend only on the material. Therefore

$$A[(\mathbf{v}, \alpha), (\mathbf{v}, \alpha)] \geqslant \frac{\rho_0}{2} \int_0^{t_1} \| \dot{\mathbf{v}}(t)\|_0^2 \, dt + \frac{g_0}{4} \int_0^{t_1} \| \mathbf{v}(t)\|_1^2 \, dt + \frac{\rho_0 c_0}{4\theta_0} \int_0^{t_1} \| \alpha(t)\|_0^2 \, dt$$

$$+ \frac{t_1 \rho_0}{2} \| \dot{\mathbf{v}}(0)\|_0^2 + \frac{t_1 \rho_0 c_0}{2\theta_0} \| \alpha(0)\|_0^2 + \frac{K}{\theta_0} \int_0^{t_1} \int_0^t \| \alpha(\tau)\|_1^2 \, d\tau \, dt$$

$$\geqslant \min \left(\frac{\rho_0}{2}, \frac{g_0}{4}, \frac{\rho_0 c_0}{4\theta_0}, \frac{K}{\theta_0} \right) |||(\mathbf{v}, \alpha)|||^2.$$

Thus, the required result follows by setting $C_A = \min(\rho_0/2, g_0/4, \rho_0 c_0/4\theta_0, K/\theta_0)$, and C_A will depend only on the material.

LEMMA 4.2. *Let* $(\bar{u}, \bar{\theta})$ *be a generalized solution of* (3.22), (3.24), (3.25) *in* $B \times [0, t_0]$ *with initial conditions*

$$\left(\overset{(l+1)}{\mathbf{u}_I}, \overset{(l+2)}{\mathbf{u}_I}, \overset{(l+1)}{\theta_I} \right)$$

and supply term $(\mathbf{b}_{l+1}, r_{l+1})$. *Then*

$$(\hat{\mathbf{u}}, \hat{\theta}) = \left(\overset{(l)}{\mathbf{u}_I} + \int_0^t \bar{\mathbf{u}}(\tau) \, d\tau, \ \overset{(l)}{\theta_I} + \int_0^t \bar{\theta}(\tau) \, d\tau \right)$$

is also a generalized solution in $B \times [0, t_0]$ *with initial conditions*

$$\left(\overset{(l)}{\mathbf{u}_I}, \overset{(l+1)}{\mathbf{u}_I}, \overset{(l)}{\theta_I} \right)$$

and supply term $(\mathbf{b}_l, r_l)$. *The pair* $(\overset{(l)}{\mathbf{u}_I}, \overset{(l)}{\theta_I})$ *is given as a solution of the system*[8]

$$\int_B \nabla\beta \cdot \mathbf{K} \nabla \overset{(l)}{\theta_I} \, d\mathbf{x} = \int_B [r_l'(0) - \theta_0 \mathbf{L}(0) \cdot \nabla \overset{(l+1)}{\mathbf{u}_I} - \rho c(0) \overset{(l+1)}{\theta_I}] \, \beta \, d\mathbf{x} \qquad \forall \beta \in \mathring{W}_2^1(B), \tag{4.17}$$

$$\int_B \nabla \mathbf{w} \cdot \mathbf{G}(0) \nabla \overset{(l)}{\mathbf{u}_I} \, d\mathbf{x} = \int_B [\overset{(l)}{\theta_I} \mathbf{L}(0) \cdot \nabla \mathbf{w} - \rho \mathbf{w} \cdot \overset{(l+2)}{\mathbf{u}_I} + \mathbf{b}_l(0) \cdot \mathbf{w}] \, d\mathbf{x} \qquad \forall \mathbf{w} \in \mathring{W}_2^1(B),$$

[8] The existence and uniqueness of the solutions to the system (4.17) is a consequence of (4.6), (4.5), and the Lax–Milgram theorem [15].

where $r_l'(0) \in L_2(B)$ *and* $\mathbf{b}_l(0) \in \mathbf{H}(B)$ *are assumed to be known,* $r_l'(0)$ *being related to the value of* r_l *at* $t = 0$ *by*

$$r_l(0) = \theta_0 \mathbf{L}(0) \cdot \nabla \overset{(l)}{\mathbf{u}}_I + \rho \dot{c}(0) \overset{(l)}{\theta}_I + r_l'(0). \tag{4.18}$$

The supply term $(\mathbf{b}_l, r_l)$ *is related to* $(\mathbf{b}_{l+1}, r_{l+1})$ *by means of the equations*

$$\begin{aligned}\int_B \mathbf{w} \cdot \mathbf{b}_{l+1}(t)\, d\mathbf{x} &= \int_B [\mathbf{w} \cdot \dot{\mathbf{b}}_l(t) - \nabla \mathbf{w} \cdot \dot{\mathbf{G}}(t) \nabla \overset{(l)}{\mathbf{u}}_I + \overset{(l)}{\theta}_I \mathbf{L}(t) \cdot \nabla \mathbf{w}]\, d\mathbf{x} \\ &\qquad \forall\, \mathbf{w} \in \mathring{W}_2{}^1(B), \\ \int_B \frac{\beta}{\theta_0} r_{l+1}(t)\, d\mathbf{x} &= \int_B \left\{ \frac{1}{\theta_0} \dot{r}_l(t) - \frac{\partial}{\partial t} \left[\dot{\mathbf{L}}(t) \cdot \nabla \overset{(l)}{\mathbf{u}}_I + \frac{\rho}{\theta_0} \dot{c}(t) \overset{(l)}{\theta}_I \right] \right\} \beta\, d\mathbf{x} \\ &\qquad \forall\, \beta \in L_2(B).\end{aligned} \tag{4.19}$$

Proof. Since $(\bar{\mathbf{u}}, \bar{\theta})$ is a generalized solution with the initial conditions and supply term stated, the pair $(\hat{\mathbf{u}}, \hat{\theta})$ will satisfy the equation

$$A[(\dot{\hat{\mathbf{u}}}, \dot{\hat{\theta}}), (\mathbf{v}, \alpha)] = S[(\mathbf{b}_{l+1}, r_{l+1}), (\mathbf{v}, \alpha)] + I[(\overset{(l+1)}{\mathbf{u}}_I, \overset{(l+2)}{\mathbf{u}}_I, \overset{(l+1)}{\theta}_I), (\mathbf{v}, \alpha)]$$

for all $(\mathbf{v}, \alpha) \in T_{t_0}$ and, in particular, for a test function with $\dot{\mathbf{v}}(t_0) = \mathbf{0}$ and $\alpha(t_0) = 0$. Then, integrating once more by parts with respect to time, using (4.19), and choosing later

$$\mathbf{v}(t) = \int_0^t \frac{1}{(\tau_1 - t_0)} \int_{\tau_1}^{t_0} (\tau - t_0)\, \dot{\mathbf{v}}'(\tau)\, d\tau\, d\tau_1,$$

$$\alpha(t) = \frac{1}{(t - t_0)} \int_t^{t_0} (\tau - t_0)\, \alpha'(\tau)\, d\tau,$$

where $(\mathbf{v}', \alpha') \in T_{t_0}$, a lengthy but straightforward calculation, at the end of which we make use of (4.17) and (4.18), show that $(\hat{\mathbf{u}}, \hat{\theta})$ satisfies the equation

$$A[(\hat{\mathbf{u}}, \hat{\theta}), (\mathbf{v}', \alpha')] = I[(\overset{(l)}{\mathbf{u}}_I, \overset{(l+1)}{\mathbf{u}}_I, \overset{(l)}{\theta}_I), (\mathbf{v}', \alpha')] + S[(\mathbf{b}_l, r_l), (\mathbf{v}', \alpha')]$$

and the lemma is proved.

THEOREM 4.1. *If there is a generalized solution to the problem defined by the equations* (3.22), (3.24), (3.25) *in* $B \times [0, t_0]$ *with given initial conditions and supply term, then this solution is unique.*

Proof. To establish uniqueness it will suffice to show that if $(\mathbf{u}, \theta)$ is a generalized solution of (3.22), (3.24), (3.25) in $B \times [0, t_0]$ with $\mathbf{b}_0(t) \equiv \mathbf{0}$, $r_0(t) \equiv 0$, $\dot{\mathbf{u}}_I(\mathbf{x}) = \mathbf{0}$, $\mathbf{u}_I(\mathbf{x}) = \mathbf{0}$, $\theta_I(\mathbf{x}) = 0$, then $(\mathbf{u}, \theta)$ is zero.

Let t_1 be chosen as in Lemma 4.1, and let us set

$$\bar{\mathbf{u}}(t) = \int_0^t \mathbf{u}(\tau)\, d\tau, \qquad \bar{\theta}(t) = \int_0^t \theta(\tau)\, d\tau.$$

Then, a simple application of Lemma 4.2 shows that $(\bar{\mathbf{u}}, \bar{\theta})$ satisfies $A[(\bar{\mathbf{u}}, \bar{\theta}), (\mathbf{v}, \alpha)] = 0$ for all $(\mathbf{v}, \alpha)$ in the set T_{t_1}. But we can extend this equation by continuity for $(\mathbf{v}, \alpha)$ in the completion of T_{t_1} under the norm

$$\left\{\int_0^{t_1} \left[\| \mathbf{v}(t)\|_1^2 + \| \dot{\mathbf{v}}(t)\|_1^2 + \| \ddot{\mathbf{v}}(t)\|_0^2 + \| \alpha(t)\|_0^2 + \| \dot{\alpha}(t)\|_0^2 + \int_0^t \| \alpha(\tau)\|_1^2\, d\tau\right] dt\right\}^{1/2}.$$

Hence, in particular, we shall have $A[(\bar{\mathbf{u}}, \bar{\theta}), (\bar{\mathbf{u}}, \bar{\theta})] = 0$. Therefore, inequality (4.16) of Lemma 4.1 will imply $\bar{\mathbf{u}}(t) = \mathbf{0}$, $\bar{\theta}(t) = 0$ for $t \in [0, t_1]$. If t_1 is in $(0, t_0]$ the repetition of the argument for the consecutive intervals $[t_1, 2t_1]$, $[2t_1, 3t_1]$, etc., proves that $\bar{\mathbf{u}}(t) = \mathbf{0}$, $\bar{\theta}(t) = 0$ for $t \in [0, t_0]$, and the result

$$\mathbf{u}(t) = \mathbf{0}, \qquad \theta(t) = 0, \qquad t \in [0, t_0]$$

follows.

THEOREM 4.2. *There exists a generalized solution* $(\mathbf{u}, \theta)$ *of Eqs.* (3.22), (3.24), (3.25) *in* $B \times [0, t_0]$ *with given initial conditions* $(\mathbf{u}_I, \dot{\mathbf{u}}_I, \theta_I) \in \mathring{\mathbf{W}}_2^1(B) \times \mathbf{L}_2(B) \times L_2(B)$ *and supply term* $(\mathbf{b}_0, r_0) \in L_2([0, t_0]; \mathbf{H}(B) \times L_2(B))$ *with* $\dot{\mathbf{b}}_0(t) \in L_2([0, t_0]; \mathbf{H}(B))$.

Proof. We use a standard argument presented, for instance, in [9, 10] and based crucially on Lemma 4.1.

Again, we choose t_1 as in Lemma 4.1 and prove the theorem for $[0, t_1]$. If t_1 is in $(0, t_0]$, we repeat the argument for consecutive intervals, thus getting the required result for $[0, t_0]$.

Since $A[(\mathbf{u}, \theta), (\mathbf{v}, \alpha)]$ is a continuous linear functional on P_{t_1}, the Riesz–Fischer theorem asserts that we may represent it in the form

$$A[(\mathbf{u}, \theta), (\mathbf{v}, \alpha)] = \langle(\mathbf{u}, \theta), f(\mathbf{v}, \alpha)\rangle_{P_{t_1}} \qquad \forall\, (\mathbf{u}, \theta) \in P_{t_1},$$

where $f\colon T_{t_1} \to P_{t_1}$ is a linear map.

If we assume that $f(\mathbf{v}, \alpha) = 0$ for some $(\mathbf{v}, \alpha) \in T_{t_1}$, we have $A[(\mathbf{u}, \theta), (\mathbf{v}, \alpha)] = 0$ for all $(\mathbf{u}, \theta) \in P_{t_1}$ and, in particular, $A[(\mathbf{v}, \alpha), (\mathbf{v}, \alpha)] = 0$. Hence, by Lemma 4.1, we have $(\mathbf{v}, \alpha) = (\mathbf{0}, 0)$ and the map f is injective; whence f^{-1}, its inverse, is a well-defined mapping of the range of f onto T_{t_1}.

On the other hand, by Lemma 4.1 and the Cauchy–Schwarz inequality, we have

$$C_A\, |||f^{-1}(\mathbf{u}', \theta')||| \leqslant \|(\mathbf{u}', \theta')\|.$$

Moreover, according to the uniqueness Theorem 4.1, the assertion

$$\langle(\mathbf{u}, \theta), f(\mathbf{v}, \alpha)\rangle_{P_{t_1}} = 0 \qquad \text{for all } (\mathbf{v}, \alpha)$$

implies $(\mathbf{u}, \theta) = (\mathbf{0}, 0)$ and the orthogonal complement of the domain of f^{-1} is the set $\{(\mathbf{0}, 0)\}$. Therefore, the domain of f^{-1} is a dense subset of P_{t_1} and, extending by continuity, f^{-1} becomes a continuous map $f^{-1}: P_{t_1} \to T'_{t_1}$, where T'_{t_1} is the completion of T_{t_1} under the norm $|||\cdot|||$.

Now, since $\mathbf{v}(0) = \mathbf{0}$, we have

$$-\int_0^{t_1}\int_B (t - t_1)\, \dot{\mathbf{v}}(t) \cdot \mathbf{b}_0(t)\, d\mathbf{x}\, dt$$

$$= \int_0^{t_1}\int_B \mathbf{v}(t) \cdot \mathbf{b}_0(t)\, d\mathbf{x}\, dt + \int_0^{t_1}\int_B (t - t_1)\, \mathbf{v}(t) \cdot \dot{\mathbf{b}}_0(t)\, d\mathbf{x}\, dt$$

and, applying the Cauch–Schwarz inequality, it follows that

$$\left| \int_0^{t_1}\int_B (t - t_1) \left[-\dot{\mathbf{v}}(t) \cdot \mathbf{b}_0(t) - \frac{\alpha(t)}{\theta_0} r_0(t)\right] d\mathbf{x}\, dt \right|$$

$$\leqslant \max\left[(1 + t_1), \frac{t_1}{\theta_0}\right] \left[\int_0^{t_1} [\|\mathbf{b}_0(t)\|_{-1}^2 + \|\dot{\mathbf{b}}_0(t)\|_{-1}^2 + \|r_0(t)\|_0^2]\, dt\right]^{1/2} |||(\mathbf{v}, \alpha)|||\,.$$

Also,

$$\left| t_1 \int_B \left[\rho \dot{\mathbf{u}}_I \cdot \dot{\mathbf{v}}(0) + \alpha(0)\, \mathbf{L}(0) \cdot \nabla \mathbf{u}_I + \rho \frac{c(0)}{\theta_0} \theta_I \alpha(0)\right] d\mathbf{x} \right|$$

$$\leqslant 2(t_1)^{1/2} \max\left(\rho_1 \|\dot{\mathbf{u}}_I\|_0\,, \|\mathbf{L}(0)\|\, \|\mathbf{u}_I\|_1 + \frac{\rho_1}{\theta_0} \|c(0)\|\, \|\theta_I\|_0\right) |||(\mathbf{v}, \alpha)|||\,.$$

Hence, $I[(\mathbf{u}_I, \dot{\mathbf{u}}_I, \theta_I), (\mathbf{v}, \alpha)] + S[(\mathbf{b}_0, r_0), (\mathbf{v}, \alpha)]$ can be considered as a continuous linear functional on T'_{t_1}. Consequently,

$$(I + S)(\mathbf{v}, \alpha) = \langle(\bar{\mathbf{v}}, \bar{\alpha}), (\mathbf{v}, \alpha)\rangle_{T_{t_1}}\,,$$

where $(\bar{\mathbf{v}}, \bar{\alpha}) \in T'_{t_1}$. Then,

$$A[(\mathbf{u}, \theta), (\mathbf{v}, \alpha)] = \langle(\mathbf{u}, \theta), f(\mathbf{v}, \alpha)\rangle_{P_{t_1}} = \langle(\bar{\mathbf{v}}, \bar{\alpha}), (\mathbf{v}, \alpha)\rangle_{T_{t_1}}$$

$$= \langle(\bar{\mathbf{v}}, \bar{\alpha}), f^{-1}(\mathbf{u}', \theta')\rangle_{T_{t_1}}$$

for all $(\mathbf{u}', \theta') \in P_{t_1}$, and if f^{-1*} is the adjoint map of f^{-1}, we have

$$\langle(\mathbf{u}, \theta), (\mathbf{u}', \theta')\rangle_{P_{t_1}} = \langle f^{-1*}(\bar{\mathbf{v}}, \bar{\alpha}), (\mathbf{u}', \theta')\rangle_{P_{t_1}} \qquad \text{for all } (\mathbf{u}', \theta') \in P_{t_1}\,.$$

From here, it follows that the solution we are searching for is given by $(\mathbf{u}, \theta) = f^{-1*}(\bar{\mathbf{v}}, \bar{\alpha})$ and the theorem is proved.

Now we turn to proving a theorem concerning the smoothness of solutions under defined conditions on the initial histories of displacement and temperature.

DEFINITION 4.5. We denote by $\mathscr{H}_n$ the Banach space completion of the set of function pairs $(\mathbf{u}_1, \theta_1) \in C^n((-\infty, 0]; \mathring{W}_2^1(B)) \cap C^{n+1}((-\infty, 0]; \boldsymbol{L}_2(B)) \times C^n((-\infty, 0]; \mathring{W}_2^1(B))$ by means of the norm

$$|(\mathbf{u}_1, \theta_1)|_n = \sum_{l=0}^{n} \| \overset{(l)}{\mathbf{u}_1}(0)\|_1 + \sum_{l=0}^{n+1} \| \overset{(l)}{\mathbf{u}_1}(0)\|_0 + \sum_{l=0}^{n} \| \overset{(l)}{\theta_1}(0)\|_0$$

$$+ \sum_{l=0}^{n} \left[\int_{-\infty}^{0} i(-\tau) \| \overset{(l)}{\mathbf{u}_1}(\tau)\|_1^2 \, d\tau\right]^{1/2} + \sum_{l=0}^{n+1} \left[\int_{-\infty}^{0} i(-\tau) \| \overset{(l)}{\mathbf{u}_1}(\tau)\|_0^2 \, d\tau\right]^{1/2}$$

$$+ \sum_{l=0}^{n} \left[\int_{-\infty}^{0} i(-\tau) \| \overset{(l)}{\theta_1}(\tau)\|_0^2 \, d\tau\right]^{1/2}.$$

THEOREM 4.3. *Suppose that the initial history* $(\mathbf{u}_1, \theta_1)$ *is in* $\mathscr{H}_n$. *Then,*

(a) *There exists a unique generalized solution* $(\mathbf{u}, \theta)$ *of Eqs.* (3.17), (3.18), (3.19), (3.20), (3.24), (3.25) *in* $B \times [0, t_0]$ *which satisfies conditions* (3.21), *and such that*

$$(\mathbf{u}, \theta) \in C^n([0, t_0]; \mathring{W}_2^1(B)) \cap C^{n+1}([0, t_0]; \boldsymbol{L}_2(B)) \times C^n([0, t_0]; L_2(B)). \quad (4.20)$$

(b) *This solution satisfies the estimate*

$$\sum_{l=0}^{n} \sup_{[0,t_0]} \| \overset{(l)}{\mathbf{u}}(t)\|_1 + \sum_{l=0}^{n+1} \sup_{[0,t_0]} \| \overset{(l)}{\mathbf{u}}(t)\|_0 + \sum_{l=0}^{n} \sup_{[0,t_0]} \| \overset{(l)}{\theta}(t)\|_0 \leqslant M\, |(\mathbf{u}_1, \theta_1)|_n, \quad (4.21)$$

where M *depends on the material and* t_0 *but does not depend on* $(\mathbf{u}_1, \theta_1)$.

Proof. We shall prove the theorem assuming that $(\mathbf{u}_1, \theta_1) \in \mathscr{H}_{n+1}$. The denseness of $\mathscr{H}_{n+1}$ in $\mathscr{H}_n$ will do the rest.

(a) Obviously, the fields

$$\overset{(l)}{\mathbf{u}_I}, \quad \overset{(l+1)}{\mathbf{u}_I}, \quad \overset{(l)}{\theta_I}$$

of Lemma 4.2 are given by

$$\overset{(l)}{\mathbf{u}_I} = \overset{(l)}{\mathbf{u}_1}(0) \quad (l = 0, 1,..., n + 2), \qquad \text{and} \qquad \overset{(l)}{\theta_I} = \overset{(l)}{\theta_1}(0) \quad (l = 0, 1,..., n + 1).$$

A straightforward computation in which we use the fact $\lim_{s\to\infty} \|\dot{\mathbf{G}}(s)\| = \lim_{s\to\infty} \|\dot{c}(s)\| = \lim_{s\to\infty} \|\dot{\mathbf{L}}(s)\| = 0$, implied by (4.7), (4.9), shows that the fields $\mathbf{b}_l(t)$, $r_l(t)$ in the same lemma are given by

$$\int_B \mathbf{w}\cdot\mathbf{b}_l(t)\,d\mathbf{x} = \int_{-\infty}^0\int_B [-\nabla\mathbf{w}\cdot\dot{\mathbf{G}}(t-\tau)\nabla\overset{(l)}{\mathbf{u}}_1(\tau) + \overset{(l)}{\theta}_1(\tau)\,\dot{\mathbf{L}}(t-\tau)\cdot\nabla\mathbf{w}]\,d\tau\,d\mathbf{x},$$

$$\int_B \frac{\beta}{\theta_0} r_l(t)\,d\mathbf{x} = -\int_{-\infty}^0\int_B \left[\ddot{\mathbf{L}}(t-\tau)\cdot\nabla\overset{(l)}{\mathbf{u}}_1(\tau) + \frac{\rho}{\theta_0}\overset{(l)}{\theta}_1(\tau)\,\ddot{c}(t-\tau)\right]\beta\,d\tau\,d\mathbf{x} \tag{4.22}$$

for all $\mathbf{w}\in \overset{\circ}{W}_2{}^1(B)$, $\beta\in L_2(B)$ and $l = 0, 1,\ldots, n+1$.

On the other hand, using (4.7) for $i = 1$, and (4.9), together with Hölder's inequality, we have

$$\begin{aligned}
&\|\mathbf{b}_l(t)\|_{-1}\\
&\leqslant \int_{-\infty}^0 \left[\|\dot{\mathbf{G}}(t-\tau)\|\,\|\overset{(l)}{\mathbf{u}}_1(\tau)\|_1 + \left(\frac{\rho_1}{\theta_0}\right)^{1/2}(\|\dot{\mathbf{G}}(t-\tau)\|\,\|\dot{c}(t-\tau)\|)^{1/2}\,\|\overset{(l)}{\theta}_1(\tau)\|_0\right] d\tau\\
&\leqslant \int_{-\infty}^0 \left\{\frac{i(t-\tau)}{i(-\tau)}\left[i(-\tau)\,R\,\|\overset{(l)}{\mathbf{u}}_1(\tau)\|_1 + \left(\frac{\rho_1}{\theta_0}\right)^{1/2} Ri(-\tau)\,\|\overset{(l)}{\theta}_1(\tau)\|_0\right]\right\} d\tau\\
&\leqslant \text{constant}\times\left[\left(\int_{-\infty}^0 i(-\tau)\,\|\overset{(l)}{\mathbf{u}}_1(\tau)\|_1^2\,d\tau\right)^{1/2} + \left(\int_{-\infty}^0 i(-\tau)\,\|\overset{(l)}{\theta}_1(\tau)\|_0^2\,d\tau\right)^{1/2}\right].
\end{aligned} \tag{4.23}$$

Similarly, and using (4.7) for $i = 2$, together with (4.10), we obtain

$$\begin{aligned}
&\|\dot{\mathbf{b}}_l(t)\|_{-1}\\
&\leqslant \left[\int_{-\infty}^0 \|\ddot{\mathbf{G}}(t-\tau)\|\,\|\overset{(l)}{\mathbf{u}}_1(\tau)\|_1 + \|\ddot{\mathbf{L}}(t-\tau)\|\,\|\overset{(l)}{\theta}_1(\tau)\|_0\right] d\tau\\
&\leqslant \text{constant}\times\left[\left(\int_{-\infty}^0 i(-\tau)\,\|\overset{(l)}{\mathbf{u}}_1(\tau)\|_1^2\,d\tau\right)^{1/2} + \left(\int_{-\infty}^0 i(-\tau)\,\|\overset{(l)}{\theta}_1(\tau)\|_0^2\,d\tau\right)^{1/2}\right],
\end{aligned} \tag{4.24}$$

$$\begin{aligned}
&\|r_l(t)\|_0\\
&\leqslant \int_{-\infty}^0 \left[\|\ddot{\mathbf{L}}(t-\tau)\|\,\|\overset{(l)}{\mathbf{u}}_1(\tau)\|_1 + \frac{\rho_1}{\theta_0}\|\ddot{c}(t-\tau)\|\,\|\overset{(l)}{\theta}_1(\tau)\|_0\right] d\tau\\
&\qquad\leqslant \text{constant}\times\left[\left(\int_{-\infty}^0 i(-\tau)\,\|\overset{(l)}{\mathbf{u}}_1(\tau)\|_1^2\,d\tau\right)^{1/2} + \left(\int_{-\infty}^0 i(-\tau)\,\|\overset{(l)}{\theta}_1(\tau)\|_0^2\,d\tau\right)^{1/2}\right].
\end{aligned} \tag{4.25}$$

By virtue of the assumption $(\mathbf{u}_1, \theta_1) \in \mathscr{H}_{n+1}$, we have that

$$\left(\overset{(n+1)}{\mathbf{u}_I}, \overset{(n+2)}{\mathbf{u}_I}, \overset{(n+1)}{\theta_I} \right) \in \mathring{\boldsymbol{W}}_2^1(B) \times \boldsymbol{L}_2(B) \times L_2(B)$$

and $\mathbf{b}_{n+1} \in \boldsymbol{H}(B)$, $\dot{\mathbf{b}}_{n+1} \in \boldsymbol{H}(B)$, $r_{n+1} \in L_2(B)$. Hence, there exists a unique generalized solution, $(\bar{\mathbf{u}}, \bar{\theta})$, to the problem

$$A[(\bar{\mathbf{u}}, \bar{\theta}), (\mathbf{v}, \alpha)] = I[(\overset{(n+1)}{\mathbf{u}_I}, \overset{(n+2)}{\mathbf{u}_I}, \overset{(n+1)}{\theta_I}), (\mathbf{v}, \alpha)] + S[(\mathbf{b}_{n+1}, r_{n+1}), (\mathbf{v}, \alpha)]$$

$$\forall (\mathbf{v}, \alpha) \in T_{t_0} \tag{4.26}$$

since the assumptions of the existence and uniqueness theorems are met.

We define now

$$\begin{aligned} \mathbf{u} &\equiv \sum_{l=0}^{n} \overset{(l)}{\mathbf{u}_I} \frac{t^l}{l!} + \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \cdots \int_0^{\tau_n} \bar{\mathbf{u}}(\tau)\, d\tau, \\ \theta &\equiv \sum_{l=0}^{n} \overset{(l)}{\theta_I} \frac{t^l}{l!} + \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \cdots \int_0^{\tau_n} \bar{\theta}(\tau)\, d\tau, \end{aligned} \tag{4.27}$$

where $(\bar{\mathbf{u}}, \bar{\theta})$ is given by (4.26). An application of Lemma 4.2 proves that

$$(\hat{\mathbf{u}}, \hat{\theta}) = \left(\overset{(n)}{\mathbf{u}_I} + \int_0^t \bar{\mathbf{u}}(\tau)\, d\tau, \overset{(n)}{\theta_I} + \int_0^t \bar{\theta}(\tau)\, d\tau \right)$$

is a solution to the equation

$$A[(\hat{\mathbf{u}}, \hat{\theta}), (\mathbf{v}, \alpha)] = I[(\overset{(n)}{\mathbf{u}_I}, \overset{(n+1)}{\mathbf{u}_I}, \overset{(n)}{\theta_I}), (\mathbf{v}, \alpha)] + S[(\mathbf{b}_n, r_n), (\mathbf{v}, \alpha)]$$

for all $(\mathbf{v}, \alpha) \in T_{t_0}$, and proceeding by induction, we see that

$$(\overset{(l)}{\mathbf{u}}, \overset{(l)}{\theta})$$

satisfies

$$A((\overset{(l)}{\mathbf{u}}, \overset{(l)}{\theta}), (\mathbf{v}, \alpha)] = I[(\overset{(l)}{\mathbf{u}_I}, \overset{(l+1)}{\mathbf{u}_I}, \overset{(l)}{\theta_I}), (\mathbf{v}, \alpha)] + S[(\mathbf{b}_l, r_l), (\mathbf{v}, \alpha)] \qquad \forall (\mathbf{v}, \alpha) \in T_{t_0} \tag{4.28}$$

for $l = 0, 1, \ldots, n$. Obviously, $(\mathbf{u}, \theta)$ defined by (4.27) has the required smoothness properties (4.20).

(b) Since (4.28) can be extended by continuity for $(\mathbf{v}, \alpha)$ in the completion of T_{t_0} under the norm given by

$$\left\{\int_0^{t_0} \left[\|\mathbf{v}(t)\|_1^2 + \|\dot{\mathbf{v}}(t)\|_1^2 + \|\ddot{\mathbf{v}}(t)\|_0^2 + \|\alpha(t)\|_0^2 + \|\dot{\alpha}(t)\|_0^2 + \int_0^t \|\alpha(\tau)\|_1^2\, d\tau\right] dt + t_0 \|\dot{\mathbf{v}}(0)\|_0^2 + t_0 \|\alpha(0)\|_0^2\right\}^{1/2}$$

we can choose

$$(\overset{(l)}{\mathbf{u}} - \overset{(l)}{\mathbf{u}}_I,\ \overset{(l)}{\theta})$$

for $(\mathbf{v}, \alpha)$. Let now t be in $(0, t_0)$ and consider the equation (4.28) for $B \times [0, t]$. If we differentiate twice with respect to t, we obtain

$$\begin{aligned}
&\frac{d}{dt}\int_B \frac{1}{2}\, \rho \overset{(l+1)}{\mathbf{u}}(t) \cdot \overset{(l+1)}{\mathbf{u}}(t)\, d\mathbf{x} + \frac{d}{dt}\int_B \frac{1}{2} \nabla\overset{(l)}{\mathbf{u}}(t) \cdot \mathbf{G}(0)\, \nabla\overset{(l)}{\mathbf{u}}(t)\, d\mathbf{x} \\
&\quad + \frac{d}{dt}\int_B \frac{1}{2}\, \rho\, \frac{c(0)}{\theta_0}\, \overset{(l)}{\theta}(t)^2\, d\mathbf{x} - \frac{d}{dt}\int_0^t\!\int_B \overset{(l)}{\theta}(\tau)\, \dot{\mathbf{L}}(t-\tau) \cdot \nabla\overset{(l)}{\mathbf{u}}(t)\, d\mathbf{x}\, d\tau \\
&\quad + \frac{d}{dt}\int_0\!\int_B \nabla\overset{(l)}{\mathbf{u}}(t) \cdot \dot{\mathbf{G}}(t-\tau)\, \nabla\overset{(l)}{\mathbf{u}}(\tau)\, d\mathbf{x}\, d\tau + \frac{1}{\theta_0}\int_B \nabla\overset{(l)}{\theta}(t) \cdot \mathbf{K}\nabla\overset{(l)}{\theta}(t)\, d\mathbf{x} \\
&\quad - \int_0^t\!\int_B \nabla\overset{(l)}{\mathbf{u}}(t) \cdot \ddot{\mathbf{G}}(t-\tau)\, \nabla\overset{(l)}{\mathbf{u}}(\tau)\, d\mathbf{x}\, d\tau - \int_B \nabla\overset{(l)}{\mathbf{u}}(t) \cdot \dot{\mathbf{G}}(0)\, \nabla\overset{(l)}{\mathbf{u}}(t)\, d\mathbf{x} \\
&\quad + 2\int_B \overset{(l)}{\theta}(t)\, \dot{\mathbf{L}}(0) \cdot \nabla\overset{(l)}{\mathbf{u}}(t)\, d\mathbf{x} + \int_0^t\!\int_B \overset{(l)}{\theta}(\tau)\, \ddot{\mathbf{L}}(t-\tau) \cdot \nabla\overset{(l)}{\mathbf{u}}(t)\, d\mathbf{x}\, d\tau \\
&\quad + \int_0^t\!\int_B \overset{(l)}{\theta}(t)\, \ddot{\mathbf{L}}(t-\tau) \cdot \nabla\overset{(l)}{\mathbf{u}}(\tau)\, d\mathbf{x}\, d\tau + \int_B \frac{\rho}{\theta_0}\, \dot{c}(0)\, \overset{(l)}{\theta}(t)^2\, d\mathbf{x} \\
&\quad + \int_0^t\!\int_B \frac{\rho}{\theta_0}\, \overset{(l)}{\theta}(t)\, \ddot{c}(t-\tau)\, \overset{(l)}{\theta}(\tau)\, d\mathbf{x}\, d\tau \\
&= \int_B \left[\overset{(l+1)}{\mathbf{u}}(t) \cdot \mathbf{b}_l(t) + \frac{1}{\theta_0}\, \overset{(l)}{\theta}(t)\, r_l(t)\right] d\mathbf{x}. \qquad (4.29)
\end{aligned}$$

Integrating Eq. (4.29) between 0 and t' and writing t, λ instead of t', t, we have

$$\begin{aligned}
&\int_B \frac{1}{2}\, \rho \overset{(l+1)}{\mathbf{u}}(t) \cdot \overset{(l+1)}{\mathbf{u}}(t)\, d\mathbf{x} + \int_B \frac{1}{2} \nabla\overset{(l)}{\mathbf{u}}(t) \cdot \mathbf{G}(0)\, \nabla\overset{(l)}{\mathbf{u}}(t)\, d\mathbf{x} + \int_B \frac{1}{2}\, \rho\, \frac{c(0)}{\theta_0}\, \overset{(l)}{\theta}(t)^2\, d\mathbf{x} \\
&\quad + \frac{1}{\theta_0}\int_0^t\!\int_B \nabla\overset{(l)}{\theta}(\tau) \cdot \mathbf{K}\nabla\overset{(l)}{\theta}(\tau)\, d\mathbf{x}\, d\tau \\
&= D + E + J + Q + N, \qquad (4.30)
\end{aligned}$$

where the expressions on the right-hand side are given below. From the Cauchy–Schwarz and Young inequalities, together with (4.9), (4.10), we can deduce, in a way similar to that used in the proof of Lemma 4.1, the following estimates for D, E, F, J, Q, N:

$$
\begin{aligned}
|D| = \Big| \int_0^t \int_0^\lambda \int_B \Big[& \nabla \overset{(l)}{\mathbf{u}}(\lambda) \cdot \ddot{\mathbf{G}}(\lambda - \tau)\, \nabla \overset{(l)}{\mathbf{u}}(\tau) - \overset{(l)}{\theta}(\tau)\, \ddot{\mathbf{L}}(\lambda - \tau) \cdot \nabla \overset{(l)}{\mathbf{u}}(\lambda) \\
& - \overset{(l)}{\theta}(\lambda)\, \ddot{\mathbf{L}}(\lambda - \tau) \cdot \nabla \overset{(l)}{\mathbf{u}}(\tau) - \frac{\rho}{\theta_0}\, \ddot{c}(\lambda - \tau)\, \overset{(l)}{\theta}(\lambda)\, \overset{(l)}{\theta}(\tau) \Big]\, d\mathbf{x}\, d\lambda\, d\tau \Big| \\
\leqslant 2t_0 \Big[& \sup_{[0,t_0]} \|\ddot{\mathbf{G}}(s)\| + \frac{\rho_1}{\theta_0} \sup_{[0,t_0]} \|\ddot{c}(s)\| \Big] \int_0^t (\| \overset{(l)}{\mathbf{u}}(\tau)\|_1^2 + \| \overset{(l)}{\theta}(\tau)\|_0^2)\, d\tau
\end{aligned}
$$

and

$$
\begin{aligned}
|E| &= \Big| \int_0^t \int_B \Big[\nabla \overset{(l)}{\mathbf{u}}(\tau) \cdot \dot{\mathbf{G}}(0)\, \nabla \overset{(l)}{\mathbf{u}}(\tau) - 2\overset{(l)}{\theta}(\tau)\, \dot{\mathbf{L}}(0) \cdot \nabla \overset{(l)}{\mathbf{u}}(\tau) - \frac{\rho}{\theta_0}\, \dot{c}(0)\, \overset{(l)}{\theta}(\tau)^2 \Big]\, d\mathbf{x}\, d\tau \Big| \\
&\leqslant 2 \Big(\|\dot{\mathbf{G}}(0)\| + \frac{\rho_1}{\theta_0} \|\dot{c}(0)\| \Big) \int_0^t (\| \overset{(l)}{\mathbf{u}}(\tau)\|_1^2 + \| \overset{(l)}{\theta}(\tau)\|_0^2)\, d\tau
\end{aligned}
$$

and

$$
\begin{aligned}
|F| &= \Big| \int_B \Big[\frac{1}{2} \rho \overset{(l+1)}{\mathbf{u}}_I \cdot \overset{(l+1)}{\mathbf{u}}_I + \frac{1}{2} \nabla \overset{(l)}{\mathbf{u}}_I \cdot \mathbf{G}(0)\, \nabla \overset{(l)}{\mathbf{u}}_I + \frac{1}{2} \rho\, \frac{c(0)}{\theta_0}\, \overset{(l)}{\theta}_I{}^2 \Big]\, d\mathbf{x} \Big| \\
&\leqslant \frac{1}{2} \Big(\rho_1 + \|\mathbf{G}(0)\| + \frac{\rho_1}{\theta_0} \|c(0)\| \Big) (\| \overset{(l+1)}{\mathbf{u}}_I \|_0^2 + \| \overset{(l)}{\mathbf{u}}_I \|_1^2 + \| \overset{(l)}{\theta}_I \|_0^2)
\end{aligned}
$$

and

$$
\begin{aligned}
|J| &= \Big| \int_0^t \int_B \Big[- \overset{(l)}{\mathbf{u}}(\tau) \cdot \dot{\mathbf{b}}_l(\tau) + \frac{1}{\theta_0} \overset{(l)}{\theta}(\tau)\, r_l(\tau) \Big]\, d\mathbf{x}\, d\tau \Big| \\
&\leqslant \tfrac{1}{2} \int_0^t (\|\dot{\mathbf{b}}_l(\tau)\|_{-1}^2 + \|r_l(\tau)\|_0^2)\, d\tau + \tfrac{1}{2} \int_0^t (\| \overset{(l)}{\mathbf{u}}(\tau)\|_1^2 + \| \overset{(l)}{\theta}(\tau)\|_0^2)\, d\tau
\end{aligned}
$$

and

$$
\begin{aligned}
|Q| &= \Big| \int_0^t \int_B [\nabla \overset{(l)}{\mathbf{u}}(t) \cdot \dot{\mathbf{G}}(t - \tau)\, \nabla \overset{(l)}{\mathbf{u}}(\tau) - \overset{(l)}{\theta}(\tau)\, \dot{\mathbf{L}}(t - \tau) \cdot \nabla \overset{(l)}{\mathbf{u}}(t)]\, d\mathbf{x}\, d\tau \Big| \\
&\leqslant \frac{1}{2} \Big[\sup_{[0,t_0]} \|\dot{\mathbf{G}}(s)\| + \Big(\frac{\rho_1}{\theta_0}\Big)^{1/2} (\sup_{[0,t_0]} \|\dot{c}(s)\|)^{1/2} (\sup_{[0,t_0]} \|\dot{\mathbf{G}}(s)\|)^{1/2} \Big] \epsilon^2 \| \overset{(l)}{\mathbf{u}}(t)\|_1^2 \\
&\quad + \frac{t_0}{\epsilon^2} \Big(\sup_{[0,t_0]} \|\dot{\mathbf{G}}(s)\| + \Big(\frac{\rho_1}{\theta_0}\Big)^{1/2} (\sup_{[0,t_0]} \|\dot{c}(s)\|)^{1/2} (\sup_{[0,t_0]} \|\dot{\mathbf{G}}(s)\|)^{1/2} \Big) \\
&\quad \times \int_0^t (\| \overset{(l)}{\mathbf{u}}(\tau)\|_1^2 + \| \overset{(l)}{\theta}(\tau)\|_0^2)\, d\tau,
\end{aligned}
$$

and finally,

$$| N | = \left| \int_B [\overset{(l)}{\mathbf{u}}(t) \cdot \mathbf{b}_l(t) - \overset{(l)}{\mathbf{u}}_I \cdot \mathbf{b}_l(0)] \, d\mathbf{x} \right|$$

$$\leqslant \| \mathbf{b}_l(t) - \mathbf{b}_l(0) \|_{-1} \| \overset{(l)}{\mathbf{u}}(t) \|_1 + \| \mathbf{b}_l(0) \|_{-1} \| \overset{(l)}{\mathbf{u}}(t) - \overset{(l)}{\mathbf{u}}_I \|_1$$

$$\leqslant \left(\int_0^t \| \dot{\mathbf{b}}_l(\tau) \|_{-1} \, d\tau + \| \mathbf{b}_l(0) \|_{-1} \right) \| \overset{(l)}{\mathbf{u}}(t) \|_1 + \| \mathbf{b}_l(0) \|_{-1} \| \overset{(l)}{\mathbf{u}}_I \|_1$$

$$\leqslant \frac{1}{2\gamma^2} \left(\int_0^t \| \dot{\mathbf{b}}_l(\tau) \|_{-1} \, d\tau + \| \mathbf{b}_l(0) \|_{-1} \right)^2$$

$$+ \frac{\gamma^2}{2} \| \overset{(l)}{\mathbf{u}}(t) \|_1^2 + \frac{1}{2} \| \mathbf{b}_l(0) \|_{-1}^2 + \frac{1}{2} \| \overset{(l)}{\mathbf{u}}_I \|_1^2$$

$$\leqslant \frac{t_0}{\gamma^2} \int_0^t \| \dot{\mathbf{b}}_l(\tau) \|_{-1}^2 \, d\tau + \left(\frac{1}{\gamma^2} + \frac{1}{2} \right) \| \mathbf{b}_l(0) \|_{-1}^2 + \frac{\gamma^2}{2} \| \overset{(l)}{\mathbf{u}}(t) \|_1^2 + \frac{1}{2} \| \overset{(l)}{\mathbf{u}}_I \|_1^2 ,$$

where ϵ and γ are arbitrary constants.

Now, using conditions (4.3), (4.4), (4.5), (4.6), on the left-hand side of (4.30), together with the estimates obtained above, and choosing

$$\gamma^2 = \epsilon^2 = \frac{1}{2} g_0 \Big/ \left(\sup_{[0,t_0]} \| \dot{\mathbf{G}}(s) \| + \left(\frac{\rho_1}{\theta_0} \right)^{1/2} (\sup_{[0,t_0]} \| \dot{c}(s) \| \sup_{[0,t_0]} \| \dot{\mathbf{G}}(s) \|) + 1 \right)^{1/2}$$

we get the inequality

$$\| \overset{(l+1)}{\mathbf{u}}(t) \|_0^2 + \| \overset{(l)}{\mathbf{u}}(t) \|_1^2 + \| \overset{(l)}{\theta}(t) \|_0^2$$

$$\leqslant m \Big\{ \| \overset{(l+1)}{\mathbf{u}_I} \|_0^2 + \| \overset{(l)}{\mathbf{u}_I} \|_1^2 + \| \overset{(l)}{\theta_I} \|_0^2 + \| \mathbf{b}_l(0) \|_{-1}^2 + \int_0^t (\| \dot{\mathbf{b}}_l(\tau) \|_{-1}^2 + \| r_l(\tau) \|_0^2) \, d\tau$$

$$+ \int_0^t (\| \overset{(l+1)}{\mathbf{u}}(\tau) \|_0^2 + \| \overset{(l)}{\mathbf{u}}(\tau) \|_1^2 + \| \overset{(l)}{\theta}(\tau) \|_0^2) \, d\tau \Big\} , \tag{4.31}$$

where m depends on t_0 and the material.

The estimate (4.31) is of the form

$$w(t) \leqslant m \left\{ w(0) + \varphi(t) + \int_0^t w(\tau) \, d\tau \right\} \tag{4.32}$$

and it can be written

$$\frac{d}{dt} \left(\exp(-mt) \int_0^t w(\tau) \, d\tau \right) \leqslant m\{w(0) + \varphi(t)\}.$$

Integrating the above inequality, and using (4.32) together with the fact $d\varphi/dt > 0$, we obtain

$$w(t) \leqslant me^{mt}\{w(0) + \varphi(t)\};$$

that is,

$$\| \overset{(l+1)}{\mathbf{u}}(t)\|_0^2 + \| \overset{(l)}{\mathbf{u}}(t)\|_1^2 + \| \overset{(l)}{\theta}(t)\|_0^2 \leqslant me^{mt} \Big\{ \| \overset{(l+1)}{\mathbf{u}_I} \|_0^2 + \| \overset{(l)}{\mathbf{u}_I} \|_1^2 + \| \overset{(l)}{\theta_I} \|_0^2 + \| \mathbf{b}_l(0)\|_{-1}^2 + \int_0^t (\| \mathbf{b}_l(\tau)\|_{-1}^2 + \| r_l(\tau)\|_0^2)\, d\tau \Big\}. \tag{4.33}$$

On account of (4.23), (4.24), (4.25), inequality (4.33) yields

$$\begin{aligned} &\| \overset{(l+1)}{\mathbf{u}}(t)\|_0 + \| \overset{(l)}{\mathbf{u}}(t)\|_1 + \| \overset{(l)}{\theta}(t)\|_0 \\ &\leqslant m_1 \exp(mt_0/2) \Big\{ \| \overset{(l+1)}{\mathbf{u}_I} \|_0 + \| \overset{(l)}{\mathbf{u}_I} \|_1 + \| \overset{(l)}{\theta_I} \|_0 + \Big(\int_{-\infty}^0 i(-\tau) \, \| \overset{(l+1)}{\mathbf{u}_1}(\tau)\|_0^2 \, d\tau \Big)^{1/2} \\ &\quad + \Big(\int_{-\infty}^0 i(-\tau) \, \| \overset{(l)}{\mathbf{u}_1}(\tau)\|_1^2 \, d\tau \Big)^{1/2} + \Big(\int_{-\infty}^0 i(-\tau) \, \| \overset{(l)}{\theta_1}(\tau)\|_0^2 \, d\tau \Big)^{1/2} \Big\} \end{aligned}$$

and Eq. (4.21) follows immediately by setting $M = 2m_1 \exp(mt_0/2)$. On account of the fact that $\mathscr{H}_{n+1}$ is dense in $\mathscr{H}_n$ and the estimate (4.21), the theorem is fully proved in the general case.

5. Asymptotic Stability of Solutions

Through the use of a functional with suitable monotonicity properties [4–6] we study, in the present section, the asymptotic behavior of the generalized solutions, whose existence and smoothness have been investigated in the previous section.

In addition to the assumptions already stated in Section 4, we now make the following ones:

(a) For fixed $s > 0$,

$$\mathbf{G}(\mathbf{x}, s) = \mathbf{G}^T(\mathbf{x}, s) \qquad \text{a.e.} \quad \text{on} \qquad B. \tag{5.1}$$

(b) For any $\mathbf{v} \in \overset{\circ}{W}{}_2^1(B)$

$$\int_B \nabla\mathbf{v} \cdot \mathbf{G}(\infty)\, \nabla\mathbf{v}\, d\mathbf{x} \geqslant g_\infty \| \mathbf{v} \|_1^2, \qquad g_\infty > 0. \tag{5.2}$$

(c) For $s \in [0, \infty)$

$$\int_B \nabla \mathbf{v} \cdot \ddot{\mathbf{G}}(s) \nabla \mathbf{v} \, d\mathbf{x} \geqslant g_2(s) \| \mathbf{v} \|_1^2 \qquad \text{for all } \mathbf{v} \in \mathring{W}_2^1(B), \tag{5.3}$$

where the function $g_2(s) \geqslant 0$, and does not vanish identically in a neighborhood of $s = 0$.

(d) The function $i(s)$ in (4.7) is L_1-integrable on $(0, \infty)$, essentially positive and monotone decreasing.

(e) For $s \in [0, \infty)$

$$-\int_B \ddot{c}(s) \alpha^2 \, d\mathbf{x} \geqslant c_2(s) \| \alpha \|_0^2, \qquad c_2(s) \geqslant 0 \quad \forall \alpha \in L_2(B). \tag{5.4}$$

(f) For all $s \geqslant 0$

$$\| \ddot{\mathbf{L}}(s) \| \leqslant \left(\frac{\rho_0}{\theta_0} \right)^{1/2} [c_2(s)]^{1/2} [g_2(s)]^{1/2}, \tag{5.5}$$

where $c_2(s)$ and $g_2(s)$ are the functions appearing in (5.3), (5.4).

Remark 5.1. Since $\lim_{s \to \infty} \| \dot{\mathbf{L}}(s) \| = 0$, condition (5.5) and the Cauchy–Schwarz inequality imply

$$\| \dot{\mathbf{L}}(s) \| \leqslant \left(\frac{\rho_0}{\theta_0} \right)^{1/2} [c_1(s)]^{1/2} [g_1(s)]^{1/2}, \qquad s \in [0, \infty), \tag{5.6}$$

where

$$c_1(s) = \int_s^\infty c_2(s') \, ds' \qquad \text{and} \qquad g_1(s) = \int_s^\infty g_2(s') \, ds'. \tag{5.7}$$

Then,

$$\int_B \dot{c}(s) \alpha^2 \, d\mathbf{x} \geqslant c_1(s) \| \alpha \|_0^2 \qquad \text{for all } \alpha \in L_2(B) \tag{5.8}$$

and

$$-\int_B \nabla \mathbf{v} \cdot \dot{\mathbf{G}}(s) \nabla \mathbf{v} \, d\mathbf{x} \geqslant g_1(s) \| \mathbf{v} \|_1^2 \qquad \text{for all } \mathbf{v} \in \mathring{W}_2^1(B). \tag{5.9}$$

As a first step in the discussion of asymptotic stability of solutions, we proceed to the definition and the study of the time behavior of a certain functional which can be considered as a "free energy" of order l.

LEMMA 5.1. *Let* $(\mathbf{u}, \theta)$ *be the generalized solution of equations* (3.17), (3.18), (3.24), (3.25) *on* $B \times [0, \infty)$ *satisfying* $(\mathbf{u}, \theta) = (\mathbf{u}_1, \theta_1)$ *on* $(-\infty, 0]$, *and let* $(\mathbf{u}_1, \theta_1)$ *be in* $\mathscr{H}_n$. *Then, the functional* $\sum_l$ *defined by*

$$\Sigma_l(t) = \int_B \left[\frac{1}{2} \nabla\overset{(l)}{\mathbf{u}}(t) \cdot \mathbf{G}(\infty) \nabla\overset{(l)}{\mathbf{u}}(t) + \frac{1}{2} \rho \frac{c(0)}{\theta_0} \overset{(l)}{\theta}(t)^2 + \frac{1}{2} \rho \overset{(l+1)}{\mathbf{u}}(t) \cdot \overset{(l+1)}{\mathbf{u}}(t)\right] d\mathbf{x}$$

$$+ \int_{-\infty}^{t} \int_B \Big\{- \tfrac{1}{2}[\nabla\overset{(l)}{\mathbf{u}}(t) - \nabla\overset{(l)}{\mathbf{u}}(\tau)] \cdot \dot{\mathbf{G}}(t-\tau) [\nabla\overset{(l)}{\mathbf{u}}(t) - \nabla\overset{(l)}{\mathbf{u}}(\tau)]$$

$$- \overset{(l)}{\theta}(\tau) \dot{\mathbf{L}}(t-\tau) \cdot [\nabla\overset{(l)}{\mathbf{u}}(t) - \nabla\overset{(l)}{\mathbf{u}}(\tau)] + \frac{\rho}{2\theta_0} \dot{c}(t-\tau) \overset{(l)}{\theta}(\tau)^2\Big\} d\mathbf{x}\, d\tau \tag{5.10}$$

is a nonnegative, nonincreasing function of t for $t \in [0, \infty)$ *and* $l = 0, 1,..., n$.

Proof. (i) We first prove that $\Sigma_l(t) \geqslant 0$ for $t \in [0, \infty)$. On the one hand, we see that, on account of conditions (4.4) and (5.2), the first integral in (5.10) is nonnegative. On the other hand, and using (5.8), (5.9), (5.6), (4.3), we have that the second integral in (5.10) is bounded below by

$$\int_{-\infty}^{t} \Big[\frac{1}{2} g_1(t-\tau) \| \overset{(l)}{\mathbf{u}}(t) - \overset{(l)}{\mathbf{u}}(\tau)\|_1^2 + \frac{\rho_0}{2\theta_0} c_1(t-\tau) \| \overset{(l)}{\theta}(\tau)\|_0^2$$

$$- \| \dot{\mathbf{L}}(t-\tau)\| \| \overset{(l)}{\theta}(\tau)\|_0 \| \overset{(l)}{\mathbf{u}}(t) - \overset{(l)}{\mathbf{u}}(\tau)\|_1\Big] d\tau$$

$$\geqslant \frac{1}{2} \int_{-\infty}^{t} \left[(g_1(t-\tau))^{1/2} \| \overset{(l)}{\mathbf{u}}(t) - \overset{(l)}{\mathbf{u}}(\tau)\|_1 - \left(\frac{\rho_0}{\theta_0}\right)^{1/2} (c_1(t-\tau))^{1/2} \| \overset{(l)}{\theta}(\tau)\|_0\right]^2 d\tau$$

and the integral on the right-hand side of above inequality is nonnegative. Thus, the first part of the lemma is proved.

(ii) We now assume that $(\mathbf{u}_1, \theta_1)$ is in $\mathscr{H}_{n+1}$ and compute the derivative of $\Sigma_l(t)$, $l = 0, 1,..., n$.

The functional $\Sigma_l(t)$ can be rewritten in the following form:

$$\Sigma_l(t) = \int_B \left[\frac{1}{2} \nabla\overset{(l)}{\mathbf{u}}(t) \cdot \mathbf{G}(0) \nabla\overset{(l)}{\mathbf{u}}(t) + \frac{1}{2} \rho \frac{c(0)}{\theta_0} \overset{(l)}{\theta}(t)^2 + \frac{1}{2} \rho \overset{(l+1)}{\mathbf{u}}(t) \cdot \overset{(l+1)}{\mathbf{u}}(t)\right] d\mathbf{x}$$

$$+ \int_0^t \int_B \Big\{\nabla\overset{(l)}{\mathbf{u}}(t) \cdot \dot{\mathbf{G}}(t-\tau) \nabla\overset{(l)}{\mathbf{u}}(\tau) - \tfrac{1}{2}\nabla\overset{(l)}{\mathbf{u}}(\tau) \cdot \dot{\mathbf{G}}(t-\tau) \nabla\overset{(l)}{\mathbf{u}}(\tau)$$

$$- \overset{(l)}{\theta}(\tau) \dot{\mathbf{L}}(t-\tau) \cdot \nabla\overset{(l)}{\mathbf{u}}(t)$$

$$+ \overset{(l)}{\theta}(\tau) \dot{\mathbf{L}}(t-\tau) \cdot \nabla\overset{(l)}{\mathbf{u}}(\tau) + \frac{\rho}{2\theta_0} \dot{c}(t-\tau) \overset{(l)}{\theta}(\tau)^2\Big\} d\mathbf{x}\, d\tau$$

$$+\int_{-\infty}^{0}\int_B \Big\{\nabla\overset{(l)}{\mathbf{u}}(t)\cdot\dot{\mathbf{G}}(t-\tau)\,\nabla\overset{(l)}{\mathbf{u}}_1(\tau)-\overset{(l)}{\theta}_1(\tau)\,\dot{\mathbf{L}}(t-\tau)\cdot\nabla\overset{(l)}{\mathbf{u}}(t)$$

$$+\frac{\rho}{2\theta_0}\dot{c}(t-\tau)\,\overset{(l)}{\theta}_1(\tau)^2+\overset{(l)}{\theta}_1(\tau)\,\dot{\mathbf{L}}(t-\tau)\cdot\nabla\overset{(l)}{\mathbf{u}}_1(\tau)$$

$$-\tfrac{1}{2}\nabla\overset{(l)}{\mathbf{u}}_1(\tau)\cdot\dot{\mathbf{G}}(t-\tau)\,\nabla\overset{(l)}{\mathbf{u}}_1(\tau)\Big\}\,d\mathbf{x}\,d\tau. \tag{5.11}$$

Differentiating (5.11) with respect to t, making use of Eqs. (4.29), (4.22), and grouping terms, we obtain, after a lengthy but straightforward calculation,

$$-\dot{\Sigma}_l(t)=\tfrac{1}{2}\int_{-\infty}^{t}\int_B \Big\{[\nabla\overset{(l)}{\mathbf{u}}(t)-\nabla\overset{(l)}{\mathbf{u}}(\tau)]\cdot\ddot{\mathbf{G}}(t-\tau)\,[\nabla\overset{(l)}{\mathbf{u}}(t)-\nabla\overset{(l)}{\mathbf{u}}(\tau)]$$

$$-\frac{\rho}{\theta_0}\ddot{c}(t-\tau)\,[\overset{(l)}{\theta}(t)-\overset{(l)}{\theta}(\tau)]^2$$

$$-2[\overset{(l)}{\theta}(t)-\overset{(l)}{\theta}(\tau)]\,\ddot{\mathbf{L}}(t-\tau)\cdot[\nabla\overset{(l)}{\mathbf{u}}(t)-\nabla\overset{(l)}{\mathbf{u}}(\tau)]\Big\}\,d\mathbf{x}\,d\tau$$

$$-\frac{1}{\theta_0}\int_B \nabla\overset{(l)}{\theta}(t)\cdot\mathbf{K}\nabla\overset{(l)}{\theta}(t)\,d\mathbf{x}. \tag{5.12}$$

Then, using (5.3), (5.4), (5.5), we have that the first double integral in (5.12) is bounded below by the expression

$$\int_{-\infty}^{t}\left[(g_2(t-\tau))^{1/2}\,\|\overset{(l)}{\mathbf{u}}(t)-\overset{(l)}{\mathbf{u}}(\tau)\|_1-\left(\frac{\rho_0}{\theta_0}\right)^{1/2}(c_2(t-\tau))^{1/2}\,\|\overset{(l)}{\theta}(t)-\overset{(l)}{\theta}(\tau)\|_0\right]^2 d\tau \tag{5.13}$$

which is nonnegative. This fact, together with assumption (4.6), yields

$$\dot{\Sigma}_l(t)\leqslant 0. \tag{5.14}$$

The denseness of $\mathscr{H}_{n+1}$ in $\mathscr{H}_n$ and estimate (4.21) imply the validity of (5.12), (5.14) for $(\mathbf{u}_1, \theta_1)$ in $\mathscr{H}_n$. Then, the proof of the lemma is complete.

We now have all the necessary means to prove the theorem concerning the behavior of solutions as $t\to\infty$.

THEOREM 5.1. *Let* $(\mathbf{u}, \theta)$ *be the generalized solution of Eqs.* (3.17), (3.18), (3.24), (3.25) *on* $B\times[0,\infty)$ *which satisfies* $(\mathbf{u},\theta)=(\mathbf{u}_1,\theta_1)$ *on* $(-\infty, 0]$, *and let* $(\mathbf{u}_1,\theta_1)$ *be in* $\mathscr{H}_n$. *Then, for* $l=0, 1,\ldots, n$

$$\|\overset{(l)}{\mathbf{u}}(t)\|_1\to 0,\qquad \|\overset{(l+1)}{\mathbf{u}}(t)\|_0\to 0,\qquad \|\overset{(l)}{\theta}(t)\|_0\to 0 \tag{5.15}$$

as $t\to\infty$.

Proof. We start by making the stronger assumption $(\mathbf{u}_1, \theta_1) \in \mathscr{H}_{n+2}$. Differentiating twice with respect to t and using integration by parts, we obtain

$$\ddot{\Sigma}_l(t) = 2\Sigma_{l+1}(t) + \int_B \Big[\nabla \overset{(l+2)}{\mathbf{u}}(t) \cdot \mathbf{G}(\infty) \nabla \overset{(l)}{\mathbf{u}}(t) + \rho \frac{c(\infty)}{\theta_0} \overset{(l)}{\theta}(t) \overset{(l+2)}{\theta}(t) + \rho \overset{(l+1)}{\mathbf{u}}(t) \cdot \overset{(l+3)}{\mathbf{u}}(t)\Big] \, d\mathbf{x}$$
$$+ \int_{-\infty}^t \int_B \Big\{ -[\nabla \overset{(l+2)}{\mathbf{u}}(t) - \nabla \overset{(l+2)}{\mathbf{u}}(\tau)] \cdot \dot{\mathbf{G}}(t-\tau) [\nabla \overset{(l)}{\mathbf{u}}(t) - \nabla \overset{(l)}{\mathbf{u}}(\tau)]$$
$$- \overset{(l+2)}{\theta}(\tau) \dot{\mathbf{L}}(t-\tau) \cdot [\nabla \overset{(l)}{\mathbf{u}}(t) - \nabla \overset{(l)}{\mathbf{u}}(\tau)] - \overset{(l)}{\theta}(\tau) \dot{\mathbf{L}}(t-\tau) \cdot [\nabla \overset{(l+2)}{\mathbf{u}}(t) - \nabla \overset{(l+2)}{\mathbf{u}}(\tau)]$$
$$+ \frac{\rho}{\theta_0} \dot{c}(t-\tau) [\overset{(l)}{\theta}(\tau) \overset{(l+2)}{\theta}(\tau) - \overset{(l)}{\theta}(t) \overset{(l+2)}{\theta}(t)] \Big\} \, d\mathbf{x} \, d\tau. \tag{5.16}$$

We denote by Z all the terms on the right-hand side of (5.16) beyond the first. On account of the fact that $\Sigma_l(t) \geqslant 0$, $l = 0, 1, \ldots, n+2$, the Cauchy–Schwarz and Young inequalities, we have the following estimate for Z:

$$|Z| \leqslant 2\Sigma_l(t)^{1/2} \Sigma_{l+2}(t)^{1/2} \leqslant \Sigma_l(t) + \Sigma_{l+2}(t) \tag{5.17}$$

and, therefore, we get the inequality

$$|\ddot{\Sigma}_l(t)| \leqslant 2[\Sigma_l(t) + \Sigma_{l+1}(t) + \Sigma_{l+2}(t)].$$

But, by Lemma 5.1, $\dot{\Sigma}_{l+1}(t) \leqslant 0$, $\dot{\Sigma}_{l+2}(t) \leqslant 0$ for $l = 0, 1, \ldots, n$. Hence

$$|\ddot{\Sigma}_l(t)| \leqslant 2[\Sigma_l(0) + \Sigma_{l+1}(0) + \Sigma_{l+2}(0)] \leqslant C$$

whence

$$\dot{\Sigma}_l(t) \ddot{\Sigma}_l(t) \leqslant |\dot{\Sigma}_l(t)| \, |\ddot{\Sigma}_l(t)| \leqslant C |\dot{\Sigma}_l(t)| = -C\dot{\Sigma}_l(t),$$

namely,

$$\dot{\Sigma}_l(t) \ddot{\Sigma}_l(t) + C\dot{\Sigma}_l(t) = \frac{d}{dt}\Big[\frac{1}{2} \dot{\Sigma}_l(t)^2 + C\Sigma_l(t)\Big] \leqslant 0. \tag{5.18}$$

Since

$$\tfrac{1}{2}\dot{\Sigma}_l(t)^2 + C\Sigma_l(t) \geqslant 0, \tag{5.19}$$

(5.18) and (5.19) imply the existence of $\lim_{t\to\infty}[\frac{1}{2}\dot{\Sigma}_l(t)^2 + C\Sigma_l(t)]$. This fact, together with $\Sigma_l(t) \geqslant 0$ and $\dot{\Sigma}_l(t) \leqslant 0$, establishes the existence of $\lim_{t\to\infty} \Sigma_l(t)$ and hence of $\lim_{t\to\infty} \dot{\Sigma}_l(t)$, and this latter limit must be zero. Thus, by (5.12), we have shown that

$$\lim_{t\to\infty} \Big\{ \int_{-\infty}^t \int_B \Big(\tfrac{1}{2}[\nabla \overset{(l)}{\mathbf{u}}(t) - \nabla \overset{(l)}{\mathbf{u}}(\tau)] \cdot \ddot{\mathbf{G}}(t-\tau) [\nabla \overset{(l)}{\mathbf{u}}(t) - \nabla \overset{(l)}{\mathbf{u}}(\tau)]$$
$$- [\overset{(l)}{\theta}(t) - \overset{(l)}{\theta}(\tau)] \ddot{\mathbf{L}}(t-\tau) \cdot [\nabla \overset{(l)}{\mathbf{u}}(t) - \nabla \overset{(l)}{\mathbf{u}}(\tau)]$$
$$- \frac{\rho}{2\theta_0} \ddot{c}(t-\tau) [\overset{(l)}{\theta}(t) - \overset{(l)}{\theta}(\tau)]^2 \Big) \, d\mathbf{x} \, d\tau + \frac{1}{\theta_0} \int_B \nabla \overset{(l)}{\theta}(t) \cdot \mathbf{K} \nabla \overset{(l)}{\theta}(t) \, d\mathbf{x} \Big\} = 0. \tag{5.20}$$

We turn now to showing that (5.20) implies the required results. In view of (5.13), condition (5.20) yields

$$\lim_{t\to\infty} \int_B \overset{(l)}{\theta}(t) \cdot \mathbf{K}\nabla\overset{(l)}{\theta}(t)\, d\mathbf{x} = 0 \tag{5.21}$$

which, on account of (4.6) and Poincare's inequality, implies

$$\| \overset{(l)}{\theta}(t)\|_0 \to 0 \qquad \text{as} \qquad t \to \infty \tag{5.22}$$

and $(5.15)_3$ is satisfied.

Now, by virtue of (5.22) we have that given any $\mu > 0$ there exists a t_1, depending on μ, such that $\| \overset{(l)}{\theta}(t)\|_0^2 < \mu$ for all $t > t_1$. Then,

$$\int_{-\infty}^{t} i(t-\tau) \| \overset{(l)}{\theta}(\tau)\|_0^2\, d\tau = \int_{-\infty}^{t_1} i(t-\tau) \| \overset{(l)}{\theta}(\tau)\|_0^2\, d\tau + \int_{t_1}^{t} i(t-\tau) \| \overset{(l)}{\theta}(\tau)\|_0^2\, d\tau.$$

But

$$\begin{aligned}\int_{-\infty}^{t} & i(t-\tau) \| \overset{(l)}{\theta}(\tau)\|_0^2\, d\tau \\ &= \int_{-\infty}^{0} i(t-t_1-\tau) \| \overset{(l)}{\theta}(t_1+\tau)\|_0^2\, d\tau \leqslant \int_{-\infty}^{0} i(-\tau) \| \overset{(l)}{\theta}(t_1+\tau)\|_0^2\, d\tau\end{aligned}$$

since $i(s)$ is monotone decreasing and $t > t_1$. Obviously, the integral on the right-hand side of above inequality is finite. Now, for each τ, $i(t-t_1-\tau) \to 0$ essentially as $t \to \infty$, and from the theorem of dominated convergence it follows that we can find a t_2 large enough as to have, given any $\nu > 0$,

$$\int_{-\infty}^{t} i(t-\tau) \| \overset{(l)}{\theta}(\tau)\|_0^2\, d\tau < \frac{\nu}{2} \qquad \text{for all } t > t_2\,.$$

On the other hand,

$$\int_{t_1}^{t} i(t-\tau) \| \overset{(l)}{\theta}(\tau)\|_0^2\, d\tau \leqslant \mu \int_{t_1}^{t} i(t-\tau)\, d\tau \leqslant \mu \int_{0}^{\infty} i(s)\, ds < \frac{\nu}{2},$$

where we have chosen μ such that $\int_0^\infty i(s)\, ds < \nu/2\mu$. Therefore,

$$\lim_{t\to\infty} \int_{-\infty}^{t} i(t-\tau) \| \overset{(l)}{\theta}(\tau)\|_0^2\, d\tau = 0$$

and since by (4.7)

$$\left| \int_{-\infty}^{t} \int_B \frac{\rho}{2\theta_0} \ddot{c}(t-\tau) [\overset{(l)}{\theta}(t) - \overset{(l)}{\theta}(\tau)]^2 \, d\mathbf{x} \, d\tau \right|$$
$$\leqslant \frac{\rho_1}{2\theta_0} R \int_{-\infty}^{t} i(t-\tau) \| \overset{(l)}{\theta}(t) - \overset{(l)}{\theta}(\tau)\|_0^2 \, d\tau$$
$$\leqslant \frac{R\rho_1}{2\theta_0} \left[2 \| \overset{(l)}{\theta}(t)\|_0^2 \int_{-\infty}^{0} i(-\tau) \, d\tau + 2 \int_{-\infty}^{t} i(t-\tau) \| \overset{(l)}{\theta}(\tau)\|_0^2 \, d\tau \right]$$

we obtain

$$\lim_{t\to\infty} \left| \int_{-\infty}^{t} \int_B \frac{\rho}{2\theta_0} \ddot{c}(t-\tau) [\overset{(l)}{\theta}(t) - \overset{(l)}{\theta}(\tau)]^2 \, d\mathbf{x} \, d\tau \right| = 0. \tag{5.23}$$

Recalling (5.20), we see that in view of (5.21), (5.23), (5.3), (5.4), and (5.5), we are left with the result

$$\begin{aligned} f(t) &\equiv \int_{-\infty}^{t} g_2(t-\tau) \| \overset{(l)}{\mathbf{u}}(t) - \overset{(l)}{\mathbf{u}}(\tau)\|_1^2 \, d\tau \\ &= \int_0^{\infty} g_2(s) \| \overset{(l)}{\mathbf{u}}(t) - \overset{(l)}{\mathbf{u}}(t-s)\|_1^2 \, ds \to 0 \qquad \text{as} \qquad t \to \infty \end{aligned} \tag{5.24}$$

which we now show implies $(5.15)_2$. We have

$$\| \overset{(l+2)}{\mathbf{u}}(t)\|_1^2 \leqslant \Sigma_{l+2}(0) = Q^2,$$

where Q is a constant. But

$$\frac{d}{ds} [\overset{(l)}{\mathbf{u}}(t) - \overset{(l)}{\mathbf{u}}(t-s) - s\overset{(l+1)}{\mathbf{u}}(t)] = \overset{(l+1)}{\mathbf{u}}(t-s) - \overset{(l+1)}{\mathbf{u}}(t) = -\int_{t-s}^{t} \overset{(l+2)}{\mathbf{u}}(\lambda) \, d\lambda$$

which yields, through integration,

$$\overset{(l)}{\mathbf{u}}(t) - \overset{(l)}{\mathbf{u}}(t-s) - s\overset{(l+1)}{\mathbf{u}}(t) = -\int_0^s \int_{t-s'}^{t} \overset{(l+2)}{\mathbf{u}}(\lambda) \, d\lambda \, ds'$$

and therefore

$$\| \overset{(l)}{\mathbf{u}}(t) - \overset{(l)}{\mathbf{u}}(t-s) - s\overset{(l+1)}{\mathbf{u}}(t)\|_1 \leqslant \tfrac{1}{2} Q s^2.$$

Hence, applying Young's inequality, we have that for any $T > 0$

$$\begin{aligned} \int_0^T g_2(s) s^2 \| \overset{(l+1)}{\mathbf{u}}(t)\|_1^2 \, ds &\leqslant 2 \int_0^T g_2(s) \| \overset{(l)}{\mathbf{u}}(t) - \overset{(l)}{\mathbf{u}}(t-s)\|_1^2 \, ds + 2 \int_0^T g_2(s) \frac{Q^2}{4} s^4 \, ds \\ &\leqslant 2 \left(f(t) + \frac{Q^2}{4} \int_0^T g_2(s) s^4 \, ds \right). \end{aligned}$$

In virtue of the hypothesis on $g_2(s)$, we have $0 < \nu \leqslant g_2(s) \leqslant \mu$ for $0 \leqslant s \leqslant \gamma$. Then, for $T < \gamma$

$$\tfrac{1}{3}\nu T^3 \| \overset{(l+1)}{\mathbf{u}}(t)\|_1^2 \leqslant 2(f(t) + \tfrac{1}{20}\mu Q^2 T^5).$$

On account of (5.24), we can take $T = \{f(t)\}^{1/4}$ for t sufficiently large. Then

$$\tfrac{1}{3}\nu \| \overset{(l+1)}{\mathbf{u}}(t)\|_1^2 \leqslant 2(f(t)^{1/4} + \tfrac{1}{20}\mu Q^2 f(t)^{1/2})$$

and, since $\nu > 0$, we have

$$\| \overset{(l+1)}{\mathbf{u}}(t)\|_1^2 \to 0 \qquad \text{as} \qquad t \to \infty \tag{5.25}$$

for $l = 0, 1,..., n$.

The result (5.25) implies $(5.15)_2$ in view of Poincaré's inequality. Also, $(5.15)_1$ is satisfied for $l = 1, 2,..., n$.

Now, if we consider the pair of fields $(\hat{\mathbf{u}}, \hat{\theta})$ defined by

$$\hat{\mathbf{u}}(t) = \hat{\mathbf{u}}_I + \int_0^t \mathbf{u}(\tau)\, d\tau, \qquad \hat{\theta}(t) = \hat{\theta}_I + \int_0^t \theta(\tau)\, d\tau, \qquad t \in (-\infty, \infty)$$

and recall Lemma 4.2 and Eqs. (4.22), we see that $(\hat{\mathbf{u}}, \hat{\theta})$ is also generalized solution on $B \times [0, \infty)$ with $(\hat{\mathbf{u}}_1, \hat{\theta}_1)$ on $(-\infty, 0]$ belonging to $\mathscr{H}_{n+3}$. Obviously,

$$\overset{(l+1)}{\hat{\mathbf{u}}}(t) = \overset{(l)}{\mathbf{u}}(t) \qquad \text{and} \qquad \overset{(l+1)}{\hat{\theta}}(t) = \overset{(l)}{\theta}(t).$$

Applying the same arguments to $(\overset{(l)}{\hat{\mathbf{u}}}, \overset{(l)}{\hat{\theta}})$, we conclude that $\lim_{t\to\infty} \| \overset{(l+1)}{\hat{\mathbf{u}}}(t)\|_1 = 0$ which, in the case $l = 0$, yields

$$\lim_{t\to\infty} \| \mathbf{u}(t)\|_1 = 0$$

and $(5.15)_1$ is also satisfied for $l = 0$.

The denseness of $\mathscr{H}_{n+2}$ in $\mathscr{H}_n$ and estimate (4.21) permit the extension to the case in which $(\mathbf{u}_1, \theta_1)$ is in $\mathscr{H}_n$. Then, the proof of the theorem is complete.

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